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On the diffusion of momentum and mass by internal gravity waves

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The interaction between short internal gravity waves and a larger-scale mean flow in the ocean is analysed in the WKBJ approximation. The wave field determines the radiation-stress term in the momentum equation of the mean flow and a similar term in the buoyancy equation. The mean flow affects the propagation characteristics of the wave field. This cross-coupling is treated as a small perturbation. When relaxation effects within the wave field are considered, the mean flow induces a modulation of the wave field which is a linear functional of the spatial gradients of the mean current velocity. The effect that this modulation itself has on the mean flow can be reduced to the addition of diffusion terms to the equations for the mass and momentum balance of the mean flow. However, there is no vertical diffusion of mass and other passive properties. The diffusion coefficients depend on the frequency spectrum and the relaxation time of the internal-wave field and can be evaluated analytically. The vertical viscosity coefficient is found to be $\nu_v \approx 4 \times 10^3\,\mathrm{cm^2/s}$ and exceeds values typically used in models of the general circulation by at least two orders of magnitude.

1. Introduction

Models of large- and meso-scale oceanographic phenomena introduce eddy-viscosity and eddy-diffusivity coefficients in order to simulate the interaction with smaller-scale fields. The basic equations are averaged over a grid scale in space and a corresponding interval in time. To close these averaged equations the terms arising from the subgrid component of the motion are approximated by diffusion terms. The diffusion coefficients are treated as free parameters and are adjusted such that the models satisfactorily reproduce the observed features of the phenomenon under study. One of the main problems of large-scale oceanography is to understand under what conditions this closure scheme is appropriate; then, if it is appropriate, one has to get independent information about the magnitude of the diffusion coefficients. The process we know most about is the diffusion of mass and other passive properties. Measuring profiles of temperature and salinity and the diffusion of tracers provides considerable information about the diffusivity on various scales. Similar observational evidence is lacking for the viscosity. To resolve the smaller-scale fields in numerical models is a problem which considerably exceeds the speed and storage capacity of present-day computers. The analytical evaluation of the diffusion coefficients has failed because of the closure problem since the small-scale field is traditionally viewed as a strongly nonlinear turbulent field. However, if the small-scale field is a weakly nonlinear wave field, a closed set of equations can be derived rigorously using weak-interaction concepts (Hasselmann 1968) and an analytical evaluation of the eddy- or wave-induced diffusion coefficients becomes feasible.

There is good evidence that the small-scale fluctuations in the ocean are largely associated with internal waves, so that weak-interaction concepts may indeed be applicable in the ocean. Internal waves are weakly nonlinear and their space and time scales are smaller than those of the large- and meso-scale motions in the ocean. In order that the latter is true for the vertical co-ordinate as well, we are restricted to high-mode-number internal waves which have vertical wavelengths much smaller than the ocean depth. There is evidence from recent observations and theoretical studies that an appreciable part of the internal-wave field satisfies this condition. A survey of internal-wave observations, together with a frequency–wavenumber spectrum which is consistent with most observations, has been given by Garrett & Munk (1972, 1975).

In this paper we analyse theoretically the interaction between the high-mode-number part of the internal-wave field and a larger-scale mean flow which arises from the propagation of internal waves within the larger-scale mean flow. There are other processes by which internal waves interact with larger-scale fields (Müller & Olbers 1975): internal waves may be generated by low-frequency currents which either interact with bottom topography (Bell 1975) or break by shear instability. Internal waves may pass their momentum and energy to the mean flow by critical-layer absorption (Bretherton 1966; Jones 1967). Also, part of the energy lost by breaking internal waves may appear in the potential energy of the mean density stratification (Olbers 1976). These processes are not considered in this paper.

The equations of motion for the mean and fluctuating components of the flow can be derived for each component separately. For a fluctuating field that is a linear wave field, the problem for the fluctuating field reduces to the problem of wave propagation in a slowly varying medium. This problem can be adequately treated in the WKBJ or geometric-optics approximation, yielding wave-train or wave-group solutions. As these wave groups propagate horizontally and vertically through physical space they slowly change their amplitudes, wavenumbers and frequencies, and thereby exchange momentum and energy, but not action, with the mean flow. The conservation of wave action has to be proved explicitly since Whitham's (1965) method of the averaged Lagrangian is not applicable to our Eulerian analysis. In order that the concept of propagating groups or slowly varying trains of internal waves is consistent, the mean flow must be geostrophically balanced. The internal-wave field will be regarded as a statistically stationary and homogeneous ensemble which is completely described by its action-density spectrum. Its equation of motion is the radiation-balance equation which describes changes of the action-density spectrum along wave-group trajectories. The coupling with the mean flow enters through the propagation terms which determine these trajectories.

The mean flow is treated deterministically. The effect that the wave field, in turn, has on the mean flow is given by the divergence of the wave-induced fluxes in the momentum and buoyancy equations for the mean flow. These fluxes arise from the nonlinear terms in the basic equations and are quadratic forms of the wave amplitude. They can be evaluated by substituting, at each point in space, the local plane-wave solutions. The wave-induced source term in the potential-vorticity equation of a quasi-geostrophic mean flow is obtained by projecting the wave-induced momentum and buoyancy fluxes onto the geostrophic eigensolution.

To solve the basic coupled equations of the problem, namely, the radiation-balance equation for the wave field and the mean-flow equations with the wave-induced fluxes, we treat the coupling as a perturbation. We assume that the modulation of the internal-wave field induced by the mean flow is small compared with the unperturbed state of the internal-wave field in the absence of the mean flow. The modulation can then be determined by a perturbation expansion of the radiation-balance equation. It is essential for the analysis that relaxation processes, i.e. irreversible-transfer and dissipation processes, exist within the internal-wave field. A net transfer of energy between the mean flow and the wave field is caused primarily by these relaxation processes. Resonant transfer can be disregarded since the resonance condition is difficult to satisfy for typical deep-ocean conditions.

The modulation of the internal-wave field is a linear functional of the spatial gradients of the mean current velocity. This signature of the interaction is well suited for empirical tests (Frankignoul 1974, 1976). The modulation, in turn, affects the mean flow by adding diffusion terms to the mass and momentum balances for the mean flow. However, there is no vertical diffusion of buoyancy and other passive properties. The diffusion coefficients depend on the unperturbed internal-wave field and its relaxation time. Formally, they take the form of non-local operators acting on the mean flow: the diffusion fluxes at a certain point in space and time depend not only on the gradient of the mean current velocity at that point, but on all gradients in a certain neighbourhood of that point. In the limit of very small relaxation times (small compared with typical propagation times of the wave field) the diffusion operators become local and can be expressed as numbers, which are the usual diffusion coefficients. Estimates of the relaxation time from resonant wave–wave interactions (Olbers 1976) suggest a nearly local description of the diffusion process. The diffusion coefficients can be evaluated analytically. Only the relaxation time and the well-established frequency spectrum of the internal-wave field are needed. For the vertical viscosity coefficient a value $\nu_v \approx 4 \times 10^3\,\mathrm{cm^2/s}$ is found. Even with a possible uncertainty of a factor of 10, this value exceeds typical values used in numerical models by at least two orders of magnitude.

2. Equations of motion

Boussinesq approximation

We shall consider motions with horizontal length scales which are much smaller than the radius of the earth and the lateral dimensions of the ocean. The ocean can then be regarded as a plane rotating fluid which is of infinite horizontal extent, incompressible, stratified, and bounded by a free surface. The equations of motion and boundary conditions are given in the Boussinesq and f-plane approximation by

$$\partial u_i/\partial t + u_j\,\partial u_i/\partial x_j - \epsilon_{ij} f u_j - b\delta_{i3} + \partial\pi/\partial x_i = 0, \tag{2.1}$$

$$\partial b/\partial t + u_j\,\partial b/\partial x_j + N_e^2(x_3)\,u_3 = 0, \tag{2.2}$$

$$\partial u_i/\partial x_i = 0, \tag{2.3}$$

$$\partial \xi/\partial t + u_\alpha\,\partial \xi/\partial x_\alpha - u_3 = 0, \quad \text{at} \quad x_3 = \xi(x_1, x_2, t), \tag{2.4}$$

$$u_\alpha\,\partial h/\partial x_\alpha - u_3 = 0, \quad \text{at} \quad x_3 = -h_0 + h(x_1, x_2), \tag{2.5}$$

$$\pi + \pi_e = 0, \quad \text{at} \quad x_3 = \xi(x_1, x_2, t). \tag{2.6}$$

Latin indices run from 1 to 3, Greek indices are 1 or 2, and $\epsilon_{ij} = 1(-1)$ if (i, j) is an even (odd) permutation of (1, 2) and zero otherwise. The subscript e refers to the equilibrium stratification, and π denotes the kinematic pressure minus the hydrostatic pressure π_e associated with the equilibrium stratification. The notation is otherwise standard.

The equilibrium stratification will be regarded as a given external field which is maintained by processes within the general circulation (e.g. upwelling balancing vertical diffusion). Superimposed on this equilibrium state are fields which consist of slowly varying and rapidly fluctuating parts. In order to work out their interaction we reformulate the equations of motion. We describe the state of the system by the state vector $\boldsymbol{\psi} = \{\mathbf{u}, b, \xi\}$. The time evolution of this state vector is given by (2.1), (2.2) and (2.4). The pressure field π enters these equations only as a forced field which is determined at every time instant by the inhomogeneous Laplace equation

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\pi = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(u_i u_j) + \epsilon_{ij} f \frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_3} b, \tag{2.7}$$

$$\pi = -\pi_e \quad \text{at} \quad x_3 = \xi, \tag{2.8}$$

$$n_i\,\partial\pi/\partial x_i = n_i \epsilon_{ij} f u_j + b + u_i u_j\,\partial n_i/\partial x_j \quad \text{at} \quad x_3 = -h_0 + h, \tag{2.9}$$

with $\mathbf{n} = (-\partial h/\partial x_1, -\partial h/\partial x_2, 1)$ being the normal vector of the bottom. Equation (2.7) follows from the divergence of the momentum balance taking the incompressibility condition into account. Equation (2.8) is the dynamical boundary condition at the surface, and (2.9) follows from the projection of the momentum balance onto the normal vector of the bottom taking the kinematic boundary condition at the bottom into account.

We now consider solutions $\boldsymbol{\psi} = \bar{\boldsymbol{\psi}} + \boldsymbol{\psi}'$ of the equations of motion that consist of a slowly varying mean flow $\bar{\boldsymbol{\psi}}$ and a rapidly fluctuating field $\boldsymbol{\psi}'$. A bar over a symbol denotes a space and time average over dimensions which are intermediate between the scales of the mean flow and those of the fluctuating field. The pressure field, as a forced field, can be decomposed into a wave-free component π_f and a wave-induced component π_w: $\pi = \pi_f + \pi_w$. The wave-free component π_f is defined as the pressure in the absence of the fluctuating field. The wave-induced component can further be decomposed into a mean and a fluctuating part: $\pi_w = \overline{\pi_w} + \pi'_w$.

Equations of motion for the mean flow

General form. The equations of motion for the mean flow are derived by averaging (2.1), (2.2) and (2.4). We disregard interactions of the mean flow with small-scale external fields. Hence the scales of N_e and h are assumed to be equal to or greater than the scales of the mean flow. In this case the equations of motion for the mean flow take the form (Hasselmann 1971)

$$\partial\overline{u}_i/\partial t + \overline{u}_j\,\partial\overline{u}_i/\partial x_j - \epsilon_{ij}f\overline{u}_j - \overline{b}\delta_{i3} + \partial\pi_f/\partial x_i = -\partial F_{ij}/\partial x_j, \tag{2.10}$$

$$\partial\overline{b}/\partial t + \overline{u}_j\,\partial\overline{b}/\partial x_j + N_e^2\overline{u}_3 = -\partial M_j/\partial x_j, \tag{2.11}$$

$$\partial\overline{\xi}/\partial t + \overline{u}_\alpha\,\partial\overline{\xi}/\partial x_\alpha - \overline{u}_3 = -\partial D^s_\alpha/\partial x_\alpha \quad \text{at} \quad x_3 = \overline{\xi}, \tag{2.12}$$

where

$F_{ij} = \overline{u'_i u'_j} + \overline{\pi_w}\delta_{ij}$ (mean wave-induced momentum flux),

$M_j = \overline{b' u'_j}$ (mean wave-induced buoyancy flux),

$D^s_\alpha = \overline{\int_{\overline{\xi}}^{\overline{\xi}+\xi'} dx_3\, u_\alpha}$ (mean wave-induced surface mass flux).

The coupling with the fluctuating field enters through the source terms on the right-hand side of the equations. The tensor $-F_{ij}$ is also called the radiation-stress tensor. It consists of the Reynolds-stress tensor $-\overline{u'_i u'_j}$ and the mean wave-induced pressure $\overline{\pi_w}$. Since we are not concerned with specific surface effects, we disregard the mean surface mass flux. Its applications to surface waves are discussed by Hasselmann (1971). The defining equation for the wave-free component π_f of the pressure is not obtained by any averaging operation. It follows directly from the definition of π_f as the pressure in the absence of the fluctuating field, i.e. from the Laplace equation (2.7), (2.8) and (2.9) with the fluctuating field ψ' set equal to zero.

From (2.10) and (2.11) we obtain the energy equation

$$\left(\frac{\partial}{\partial t} + \overline{u}_j\frac{\partial}{\partial x_j}\right)(\tfrac{1}{2}\overline{u}_i\overline{u}_i + \tfrac{1}{2}N_e^{-2}\overline{b}^2) + \frac{\partial}{\partial x_j}(\pi_f\overline{u}_j + F_{ij}\overline{u}_i + N_e^{-2}\overline{b}M_j)$$
$$= F_{ij}\frac{\partial}{\partial x_j}\overline{u}_i + N_e^{-2}M_j\frac{\partial}{\partial x_j}\overline{b} + \overline{b}M_3\frac{\partial}{\partial x_3}N_e^{-2} + \tfrac{1}{2}\overline{b}^2\overline{u}_3\frac{\partial}{\partial x_3}N_e^{-2}. \tag{2.13}$$

The first two terms on the right-hand side describe the energy exchange with the fluctuating field. The last two terms, representing the energy exchange with the equilibrium stratification, become negligible if the mean flow is specified to be quasi-geostrophic ($\overline{u}_3 = 0$) and the fluctuating field is specified to be an internal-wave field ($M_3 = 0$).

Quasi-geostrophic mean flow. A quasi-geostrophic mean flow is completely described by its stream function $\phi = f^{-1}\pi_f$. Its equation of motion is the potential-vorticity equation. In the presence of a fluctuating field the potential-vorticity equation takes the form

$$\left(\frac{\partial}{\partial t}+\bar{u}_\alpha\frac{\partial}{\partial x_\alpha}\right)\left(\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}+\frac{\partial}{\partial x_3}\frac{f^2}{N_e^2}\frac{\partial}{\partial x_3}\right)\phi = -\epsilon_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_j}F_{\beta j} - \frac{\partial}{\partial x_3}\frac{f}{N_e^2}\frac{\partial}{\partial x_j}M_j. \quad (2.14)$$

The mean wave-induced pressure $\overline{\pi_w}$ does not contribute to the wave-induced source term in (2.14) since $\epsilon_{\alpha\beta}\,\partial^2(\overline{\pi_w}\,\delta_{\beta j})/\partial x_\alpha\,\partial x_j = 0$. If we specify the fluctuating field to be an internal-wave field the vertical component M_3 of the wave-induced buoyancy flux vanishes. The potential-vorticity equation then reduces to

$$\left(\frac{\partial}{\partial t}+\bar{u}_\alpha\frac{\partial}{\partial x_\alpha}\right)\left(\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}+\frac{\partial}{\partial x_3}\frac{f^2}{N_e^2}\frac{\partial}{\partial x_3}\right)\phi = -\epsilon_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_j}S_{\beta j}, \quad (2.15)$$

where

$$S_{\alpha j} = F_{\alpha j}-\epsilon_{\alpha\beta}\frac{f}{N_e^2}M_\beta\delta_{j3}. \quad (2.16)$$

Similarly the energy equation (2.13) reduces to

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x_\alpha}\right)^2+\tfrac{1}{2}\frac{f^2}{N_e^2}\left(\frac{\partial\phi}{\partial x_3}\right)^2\right\}+\text{div. terms} = -\epsilon_{\alpha\beta}S_{\beta j}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_j}\phi. \quad (2.17)$$

As the mean flow we envisage the meso-scale eddy motion in the open ocean. Its geostrophic balance and its characteristic scales and amplitudes have been established by the Mid-Ocean Dynamics Experiment (MODE programme) in the western North Atlantic. Typical main thermocline values are (Robinson 1975): horizontal velocity $\bar{u}_h = 5\,\text{cm/s}$; horizontal length scale $L_h = 100\,\text{km}$; vertical length scale $L_v = 1\,\text{km}$; time scale $T = 50$ days.

Equations for the fluctuating field

The equations for the fluctuating field are obtained by subtracting the equations for the mean flow from the equations for the complete flow. For the equations of motion we obtain

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)u_i'-\epsilon_{ij}fu_j'-b'\delta_{i3}+\frac{\partial}{\partial x_i}\pi_w'+u_j'\frac{\partial}{\partial x_j}\bar{u}_i = -\frac{\partial}{\partial x_j}(u_i'u_j'-\overline{u_i'u_j'}), \quad (2.18a)$$

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)b'+N_e^2u_3'+u_j'\frac{\partial}{\partial x_j}\bar{b} = -\frac{\partial}{\partial x_j}(u_j'b'-\overline{u_j'b'}), \quad (2.18b)$$

$$\left(\frac{\partial}{\partial t}+\bar{u}_\alpha\frac{\partial}{\partial x_\alpha}\right)\xi'-u_3'+u_\alpha'\frac{\partial}{\partial x_\alpha}\bar{\xi} = -\left(u_\alpha'\frac{\partial}{\partial x_\alpha}\xi'-\overline{u_\alpha'\frac{\partial}{\partial x_\alpha}\xi'}\right) \quad \text{at} \quad x_3 = \bar{\xi}+\xi'. \quad (2.18c)$$

The interaction with the mean flow enters through the advection terms $\bar{u}_j\,\partial\psi'/\partial x_j$ and $u_j'\,\partial\bar{\Psi}/\partial x_j$. The latter are usually small because they contain derivatives of the slowly varying mean flow.

The defining equation of the wave-induced pressure π_w is

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\pi_w = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(u_i'u_j')-2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(u_i'\bar{u}_j)+\epsilon_{ij}f\frac{\partial}{\partial x_i}u_j'+\frac{\partial}{\partial x_3}b', \quad (2.19a)$$

$$\pi_w = -\pi_e-\pi_f \quad \text{at} \quad x_3 = \bar{\xi}+\xi', \quad (2.19b)$$

$$n_i\frac{\partial}{\partial x_i}\pi_w = n_i\epsilon_{ij}fu_j'+b'+u_i'u_j'\frac{\partial}{\partial x_j}n_i+2\bar{u}_ju_i'\frac{\partial}{\partial x_j}n_i \quad \text{at} \quad x_3 = -h_0+h. \quad (2.19c)$$

The fluctuating part π'_w enters the momentum balance of the fluctuating field, the mean part $\overline{\pi_w}$ the momentum balance of the mean flow.

The equations of motion for the mean flow and the fluctuating field are not closed. The equations for the mean flow constitute the first equations of a hierarchy of equations which determine the time evolution of nth-order mean values. Because of the quadratic advection terms the equations for nth-order mean values always involve $(n+1)$th-order mean values. The central problem is the closure problem, which is to approximate $(n+1)$th-order mean values by lower-order mean values in order to close the otherwise infinite sequence of equations. It is well known from theories of weak and strong interactions that a rigorous closure is only possible if the fluctuating field is a weakly nonlinear wave field. In this case, the above hierarchy can be closed by the 'Gaussian assumption' (Hasselmann 1968) and a closed set of equations involving only first- and second-order mean values can be derived.

Internal waves in a slowly-varying mean flow

We now specify the fluctuating field to be a weakly nonlinear field of internal gravity waves propagating in a slowly varying mean flow. We derive their properties neglecting, for the moment, the small nonlinearities. Although there exist many papers on this problem, they are not directly applicable to our problem since they either neglect rotation or use the Lagrangian frame of reference. Linearizing (2.18) and (2.19) results in the traditional set of equations for internal gravity waves modified only by the mean-flow-induced advection terms. Since we assumed that the scales of the mean flow differ greatly from those of the fluctuating field we are restricted to high-mode-number internal waves.

The distribution of internal-wave energy over the different modes is still an open question. The first model of Garrett & Munk (1972) suggests that the energy is distributed nearly equally over the first twenty modes. The more recent model of Garrett & Munk (1975), which attributes most of the observed fine-structure to internal waves and which takes into account a larger variety of, and improvements in, measurement techniques, settles at a somewhat lower bandwidth. The most recent estimate from the IWEX array (Müller *et al.* 1975) again suggests about twenty excited modes. Also, resonant wave–wave interaction tends to transfer energy to high mode numbers (Olbers 1976). Hence there is sufficient indication that an appreciable part of the internal-wave energy is in mode numbers for which the vertical wavelength is small compared with the vertical length scale of the mean flow. Obvious exceptions to this statement are the internal tides (Hendry 1975; Wunsch 1975).

The concept of well-defined normal modes for the full water column, however, appears to be questionable since typical vertical propagation times are comparable with typical interaction times. The phases will thus be randomized before a mode can be formed. This suggests a description of the internal wave field as a superposition of wave trains or wave groups which propagate horizontally and vertically through physical space. These are plane waves when viewed on the scale of the wave field, but their amplitude wavenumber vector and frequency slowly change when viewed on the scale of the mean flow. A systematic derivation of their properties is obtained by the standard methods of geometric optics (WKBJ approximation).

The WKBJ approximation. Formally, this consists of an asymptotic expansion of the solution in powers of a small parameter ϵ. We suppose that the scales of the amplitude, wavenumber and frequency are comparable with the scales of the mean flow and are of order unity, whereas the wavelength and the wave period are $O(\epsilon)$. Hence we set

$$\{\mathbf{u}', b', \pi'_w\} = \sum_{n=0}^{\infty} \epsilon^n a\{\mathbf{U}^{(n)}, \epsilon^{-1}B^{(n)}, \Pi_w^{(n)}\} \exp(i\epsilon^{-1}\Theta) + \text{complex conjugate}, \quad (2.20)$$

where the amplitude $a(\mathbf{x}, t)$, the amplitude factors $\mathbf{U}^{(n)}(\mathbf{x}, t)$, $B^{(n)}(\mathbf{x}, t)$ and $\Pi^{(n)}(\mathbf{x}, t)$ and the phase function $\Theta(\mathbf{x}, t)$ vary on scale unity. The gradients of $\mathbf{u}'b'$ and π'_w are dominated by the local frequency and wavenumber

$$\omega = -\epsilon^{-1}\partial\Theta(\mathbf{x}, t)/\partial t, \quad k_i = \epsilon^{-1}\partial\Theta(\mathbf{x}, t)/\partial x_i, \quad (2.21)$$

which are $O(\epsilon^{-1})$. The requirement that the lowest order represents sinusoidal waves in a uniform medium defined by the local values of N_e, $\bar{\mathbf{u}}$ and $\bar{b}$ implies that f, N_e and $\bar{b}$ are $O(\epsilon^{-1})$ and $\bar{\mathbf{u}}$ is $O(1)$. This scaling of the external parameters and mean-flow quantities differs from the scaling obtained when rotation is neglected (Bretherton 1969). Also, the restrictions on the kinematics of the mean flow implied by this scaling are different. Expanding the mean flow variables according to

$$\{\bar{\mathbf{u}}, \bar{b}, \pi_f\} = \sum_{n=0}^{\infty} \epsilon^n\{\bar{\mathbf{u}}^{(n)}, \epsilon^{-1}\bar{b}^{(n)}, \epsilon^{-1}\pi_f^{(n)}\}, \quad (2.22)$$

we find from the horizontal momentum balance of the mean flow that

$$-\epsilon_{\alpha\beta} f \bar{u}_\beta^{(0)} + \partial\pi_f^{(0)}/\partial x_\alpha = 0, \quad (2.23)$$

and from the vertical momentum balance that

$$-\bar{b}^{(0)} + \partial\pi_f^{(0)}/\partial x_3 = 0. \quad (2.24)$$

In order that the concept of propagating wave groups be a consistent one, the mean flow must be geostrophically balanced. Furthermore we find from the buoyancy equation that $\bar{u}_3^{(0)} = 0$, which is slightly more restrictive than the usual $\partial\bar{u}_\alpha^{(0)}/\partial x_\alpha = 0$. For meso-scale motions in the open ocean these constraints represent valid approximations.

Local relations. The zeroth order of the WKBJ approximation determines the dispersion relation

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t) = \omega_0 + k_i\bar{u}_i, \quad \omega_0 = \Omega_0(\mathbf{k}, \mathbf{x}, t) = k^{-1}(N_e^2 k_\alpha k_\alpha + f^2k_3^2)^{\frac{1}{2}}, \quad (2.25)$$

which relates the local frequency ω to the local wavenumber vector $\mathbf{k} = (k_1, k_2, k_3)$. Here ω_0 denotes the intrinsic frequency, which is the frequency measured by an observer moving with the mean flow. The local amplitude factors

$$\begin{Bmatrix} U_1^{(0)} \\ U_2^{(0)} \\ U_3^{(0)} \\ B^{(0)} \\ \Pi_w^{(0)} \end{Bmatrix} = \left(\frac{k_\alpha k_\alpha}{k_3^2}\frac{1}{k^2}\right)^{\frac{1}{2}} \begin{Bmatrix} k_3^2(k_1 + ifk_2/\omega_0)/k_\alpha k_\alpha, \\ k_3^2(k_2 - ifk_1/\omega_0)/k_\alpha k_\alpha, \\ -k_3, \\ ik_3 N_e^2/\omega_0, \\ (N_e^2 - \omega_0^2)/\omega_0, \end{Bmatrix} \quad (2.26)$$

determine the relative amplitudes of the field variables. The mean flow affects the local wave solution only through the Doppler shift term $k_i \bar{u}_i$ in the dispersion relation. The amplitude factors are independent of the mean flow. Inertial oscillations ($\omega_0 = f$, $k_{1,2} = 0$) are not affected by a quasi-geostrophic mean flow since $\bar{u}_3 = 0$.

Propagation equations. Changes in the position $\mathbf{x}$ and the wavenumber $\mathbf{k}$ of a wave group can be inferred from the propagation equations

$$\left.\begin{aligned} v_i &= dx_i/dt = \partial\Omega/\partial k_i = v_i^0 + \bar{u}_i, \\ r_i &= dk_i/dt = -\partial\Omega/\partial x_i = r_i^0 - k_j\,\partial\bar{u}_j/\partial x_i. \end{aligned}\right\} \tag{2.27}$$

The group velocity v_i and the rate of refraction r_i consist of two parts, one due to the unperturbed frequency Ω_0 and one to the Doppler shift $k_i \bar{u}_i$. The propagation equations can be solved if initial values for the position and wavenumber are given. They describe the trajectories of wave groups in the phase space $\{\mathbf{x}, \mathbf{k}\}$. The propagation equations are purely kinematical. Their derivation uses only the existence of a dispersion relation and the definition of ω and $\mathbf{k}$ as derivatives of a phase function. It follows from (2.25) and (2.27) that

$$d\omega/dt = \partial\Omega/\partial t. \tag{2.28}$$

The intrinsic frequency changes according to

$$d\omega_0/dt = -k_i v_j^0\,\partial\bar{u}_i/\partial x_j + \partial\Omega_0/\partial t + \bar{u}_i\,\partial\Omega_0/\partial x_i. \tag{2.29}$$

The wavenumber (or frequency) of a wave group changes along its ray only if the dispersion relation depends explicitly on space (or time).

Conservation of wave action. Changes in the amplitude of a wave group along its ray (the transport equations) are determined by the condition that the first order of the WKBJ approximation has a unique solution. The transport equations determine changes in both the magnitude and phase of the amplitude. Since we shall consider an ensemble of internal waves with randomly distributed phases, we are only interested in changes of the magnitude. These can be determined very elegantly if the equations of motion can be derived from a Lagrangian density using Hamilton's principle. In this case Whitham's method of the averaged Lagrangian (Whitham 1965; Bretherton 1968) yields the conservation of wave action, from which changes along rays in the magnitude of the amplitude can be computed.

Our Eulerian equations of motion cannot be derived from a Lagrangian density. It is of course possible to study the interaction in the Lagrangian frame of reference, i.e. the interaction between internal waves and the Lagrangian mean flow (cf. Bretherton 1970). This interaction, however, differs from the Eulerian interaction since the effect of the wave field on the mean flow and the difference between the Eulerian and Lagrangian mean flows are both of quadratic order in the wave amplitude. If the Eulerian interaction is going to be studied, the cumbersome algebra of the first-order WKBJ approximation cannot be avoided. The correct result can, however, also be obtained by the following heuristic argument.

From the momentum and buoyancy balances for the internal-wave field, i.e. from the linearized version of (2.18), we obtain the energy equation

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\left(\tfrac{1}{2}\overline{u_i'u_i'}+\tfrac{1}{2}N_e^{-2}\overline{b'b'}\right)+\frac{\partial}{\partial x_i}\left(\overline{\pi_w' u_i'}\right) = -\overline{u_i'u_j'}\frac{\partial}{\partial x_j}\bar{u}_i - N_e^{-2}\overline{b'u_j'}\frac{\partial}{\partial x_j}\bar{b}+\tfrac{1}{2}\overline{b'b'}\bar{u}_3\frac{\partial}{\partial x_3}N_e^{-2}. \qquad (2.30)$$

Substituting the values of $\mathbf{u}'$, b' and π_w' appropriate to a local plane-wave solution, we find

$$\frac{\partial}{\partial t}E+\frac{\partial}{\partial x_j}(v_j E) = -\frac{E}{\omega_0}\left(f_{ij}\frac{\partial}{\partial x_j}\bar{u}_i+N_e^{-2}m_j\frac{\partial}{\partial x_j}\bar{b}-\tfrac{1}{2}q\bar{u}_3\frac{\partial}{\partial x_3}N_e^{-2}\right), \qquad (2.31)$$

where

$$f_{ij} = \omega_0\,\mathrm{Re}\{U_i^{(0)}U_j^{(0)*}\}, \quad m_j = \omega_0\,\mathrm{Re}\{B^{(0)}U_j^{(0)*}\}, \quad q = \omega_0\,\mathrm{Re}\{B^{(0)}B^{(0)*}\}, \qquad (2.32)$$

and $E = 2aa^*$ is the total energy density of the wave. Note that $m_3 = 0$ since $B^{(0)}$ and $U_3^{(0)}$ are orthogonal. The action density N of a wave is defined as the total energy density divided by the intrinsic frequency, i.e. $N = E/\omega_0$ (Bretherton & Garrett 1968). Its equation of motion is

$$\frac{\partial}{\partial t}N+\frac{\partial}{\partial x_j}(v_j N) = \frac{1}{\omega_0}\left\{\frac{\partial}{\partial t}E+\frac{\partial}{\partial x_j}(v_j E)\right\}+E\left\{\frac{\partial}{\partial t}+v_j\frac{\partial}{\partial x_j}\right\}\frac{1}{\omega_0}. \qquad (2.33)$$

Substituting (2.31) and (2.29) into (2.33), we find

$$\frac{\partial}{\partial t}N+\frac{\partial}{\partial x_j}(v_j N) = -\frac{N}{\omega_0}\left\{(f_{ij}-k_i v_j^0)\frac{\partial}{\partial x_j}\bar{u}_i+N_e^{-2}m_\alpha\frac{\partial}{\partial x_\alpha}\bar{b}-\tfrac{1}{2}q\bar{u}_3\frac{\partial}{\partial x_3}N_e^{-2}+\bar{u}_3\frac{\partial}{\partial x_3}\Omega_0\right\}. \qquad (2.34)$$

For a quasi-geostrophic mean flow, i.e. for a flow with $\bar{u}_3 = 0$ and

$$\partial\bar{b}/\partial x_\alpha = f\epsilon_{\alpha\beta}\,\partial\bar{u}_\beta/\partial x_3,$$

the source term on the right-hand side of (2.34) vanishes and the conservation of wave action is obtained:

$$\partial N/\partial t+\partial(v_j N)/\partial x_j = 0. \qquad (2.35)$$

This canonical result is presumably no accident and deserves explanation in a general theory. With the conservation of wave action, we are within a familiar framework which provides simple algebraic expressions for most of the interaction terms.

The coupled system of equations

We now generalize the results of the previous section to a statistical ensemble of internal waves and formulate the basic equations which describe the interaction between internal waves and a quasi-geostrophic mean flow.

Statistical ensemble of internal waves. Consider a superposition of locally plane internal waves

$$\{\mathbf{u}', b', \pi_w'\} = \int d^3k\, a(\mathbf{k})\{\mathbf{U}^{(0)}(\mathbf{k}), B^{(0)}(\mathbf{k})\, \Pi^{(0)}(\mathbf{k})\}\exp\left[i(k_i x_i-\omega t)\right]+\mathrm{c.c.} \qquad (2.36)$$

We are interested only in average properties of the wave field. Average will be defined as the average over a hypothetical ensemble of field realizations. If the wave field is homogeneous and stationary this average is equivalent to spatial or time averages. The amplitudes $a(\mathbf{k})$ then satisfy the orthogonality conditions

$$\left.\begin{aligned}\langle a(\mathbf{k})\,a(\mathbf{k}')\rangle &= 0,\\ \langle a(\mathbf{k})\,a^*(\mathbf{k}')\rangle &= \tfrac{1}{2}\omega_0\,n(\mathbf{k})\,\delta(\mathbf{k}-\mathbf{k}'),\end{aligned}\right\} \tag{2.37}$$

where $n(\mathbf{k})$ is the action density spectrum. The angle brackets denote ensemble averages. Internal waves propagating in a slowly varying medium are only locally homogeneous and stationary. In this case the wave field can still be described by its action density spectrum, but this now becomes a slowly varying function of space and time, $n = n(\mathbf{k};\mathbf{x},t)$. Its evolution in time is governed by the radiation-balance equation

$$\partial n/\partial t + v_i\,\partial n/\partial x_i + r_i\,\partial n/\partial k_i = S[n], \tag{2.38}$$

which follows from (2.35) and the canonical form of the propagation equations (2.27). We have added a source function S which describes the generation, transfer and dissipation of wave action by processes we have neglected so far (Müller & Olbers 1975). Resonant wave–wave interaction among internal waves which arise from the nonlinear terms in (2.18) presumably constitutes one of the major components of S (Olbers 1976). The consideration of wave–wave interaction and other irreversible processes within the internal-wave field will turn out to be of crucial importance for our analysis. The interaction with the mean flow enters the radiation-balance equation through the propagation terms $v_i\,\partial n/\partial x_i$ and $r_i\,\partial n/\partial k_i$ since v_i and r_i depend on the mean current velocity. This interaction conserves wave action.

Wave-induced source terms. The mean wave-induced source terms in the equations of motion for the mean flow are quadratic forms of the wave amplitude. They can be expressed in terms of the action-density spectrum by substituting ensemble averages for space–time averages, e.g.

$$\overline{u_i'(\mathbf{x},t)\,u_j'(\mathbf{x}+\mathbf{r},t+\tau)} = \tfrac{1}{2}\int d^3k\,\omega_0\,n\{U_i^{(0)}\,U_j^{0(*)}\exp\left[-i(k_i r_i-\omega\tau)\right]+\text{c.c.}\}. \tag{2.39}$$

Explicitly we find for the source term $S_{\alpha j}$ in the potential-vorticity equation (2.15)

$$S_{\alpha j} = \int d^3k\,nk_\alpha\,v_j^0. \tag{2.40}$$

Note that the source term has the canonical structure of wavenumber times group velocity which is expected in Lagrangian formulations. Since we are restricted to quasi-geostrophic mean motions, the potential-vorticity equation is an adequate equation to work with. However, in order to describe the interaction in terms of familiar viscosity and diffusivity coefficients we will present the analysis for the momentum and buoyancy equations. When doing so the kinematic restriction of the mean flow must be borne in mind. The source term F_{ij} in the momentum equation (2.10) involves the mean wave-induced pressure $\overline{\pi_w}$, which has to be determined from the Laplace equation (2.19). We shall neglect $\overline{\pi_w}$ since it does not affect a quasi-geostrophic mean flow. Hence we have

$$F_{ij} = \int d^3k\,nf_{ij}, \quad M_j = \int d^3k\,nm_j, \tag{2.41}$$

with f_{ij} and m_j given by (2.32).

The radiation-balance equation and the mean-flow equation with the wave-induced source terms represent the closed but coupled system of equations which describes the interaction between a statistical ensemble of internal waves and a deterministic mean flow. The interaction is physically and formally analogous to the interaction between the thermal motion of molecules and the macroscopic flow treated in statistical mechanics. The radiation-balance equation corresponds to the Boltzmann equation, the mean-flow equations correspond to the macroscopic conservation laws for mass, momentum and thermal energy.

3. Physical interpretation of the wave-induced momentum diffusion

Internal waves propagating in a larger-scale mean flow transfer energy, momentum and buoyancy. In order to understand this transfer physically, concepts such as wave energy and wave momentum need to be defined. The concept of wave energy and its flux, as already used in (2.31), provides no difficulty and is consistent with the general findings of Bretherton & Garrett (1968). The concept of wave momentum (buoyancy) and wave-induced momentum (buoyancy) flux, however, lacks a general approach, although some progress has been made for Lagrangian systems (cf. Bretherton 1970). It is straightforward to evaluate the wave-induced momentum or buoyancy flux in terms of the wavenumber, frequency and square of the amplitude. The work done by these fluxes against the mean fields correctly describes changes in the wave energy. It is not, however, clear whether a mean momentum carried with the group velocity can be associated with wave groups, nor is it clear what the physical significance of that momentum is.

Wave momentum. In our case the restriction to a quasi-geostrophic mean flow again provides us with canonical results. For a quasi-geostrophic mean flow only those components of the wave-induced momentum and buoyancy fluxes which contribute to the source term $S_{\alpha j}$ in the potential-vorticity equation are significant. For a single wave group we find $S_{\alpha j} = Nk_\alpha v_j^0$. If we attribute all wave-induced changes in the potential vorticity to the radiation stress, the horizontal momentum balance of the mean flow is given by

$$\frac{\partial}{\partial t}\overline{u}_\alpha + \frac{\partial}{\partial x_j}(\overline{u}_j\overline{u}_\alpha) - \epsilon_{\alpha\beta} f \overline{u}_\beta + \frac{\partial}{\partial x_\alpha}\pi_f = -\frac{\partial}{\partial x_j}S_{\alpha j}. \tag{3.1}$$

The mean momentum of the fluid can be approximated by $\overline{\mathbf{P}} = \overline{\rho \mathbf{u}} \approx \rho_0 \overline{\mathbf{u}}$ since the residual terms represent higher-order terms in the Boussinesq expansion $\Delta\rho/\rho_0 \rightarrow 0$ with $g\Delta\rho/\rho_0$ finite. If we decompose the horizontal momentum $\overline{P}_\alpha = P_\alpha^f + P_\alpha^w$ with $P_\alpha^w = Nk_\alpha$, (3.1) takes the form

$$\frac{\partial}{\partial t}(P_\alpha^f + P_\alpha^w) + \frac{\partial}{\partial x_j}\{\overline{u}_j P_\alpha^f + (\overline{u}_j + v_j^0) P_\alpha^w\} - \epsilon_{\alpha\beta} f \overline{u}_\beta + \frac{\partial}{\partial x_\alpha}\pi_f = 0. \tag{3.2}$$

It states that the momentum P_α^m is advected with the mean velocity $\overline{\mathbf{u}}$, whereas the momentum P_α^w is advected with the group velocity $\mathbf{v}$. Hence a horizontal momentum density

$$P_\alpha^w = Nk_\alpha \tag{3.3}$$

can be attributed to wave groups. For the following argument it need not concern us how this wave momentum is embodied in the mean flow and how it is related to the external forces which would be required to generate the wave impulsively from rest (cf. Bretherton 1969; McIntyre 1973).

Momentum flux in a shear flow. We assume that the internal-wave field can be regarded as an equilibrium state wherein external generation processes are balanced by relaxation processes. The latter processes are irreversible-transfer and dissipation processes and correspond to particle collisions in statistical mechanics. Their characteristic time scale, the relaxation time τ_R, can also be interpreted as the time a wave group can move freely before it is forced back into equilibrium by relaxation processes. Between 'collisions', the wave group is affected by the mean flow. The mean flow changes the wavenumber $\mathbf{k}$, the group velocity $\mathbf{v}(\mathbf{k})$ and the total momentum $\mathscr{P}^w_\alpha = \mathscr{N} k_\alpha$ but not the total action $\mathscr{N}$ of a wave group. Consider a mean flow with a constant vertical shear $\partial \bar{u}_1/\partial x_3$. In this case only the vertical wavenumber changes. Within a relaxation time it has changed from its equilibrium value by an amount

$$\Delta k_3 = -k_1 \tau_R \partial \bar{u}_1/\partial x_3. \tag{3.4}$$

Here we have disregarded the refraction due to the equilibrium stratification N_e. Hence two corresponding wave groups, one passing from above, $k_3 = k_3^0 \geqslant 0$, and one passing from below, $k_3 = -k_3^0$, through a horizontal plane cause a vertical flux of horizontal momentum

$$\mathscr{F}_{13} = \mathscr{N} k_1 v_3^0 (k_3^0 + \Delta k_3) + \mathscr{N} k_1 v_3^0 (-k_3^0 + \Delta k_3). \tag{3.5}$$

There is no change in the wave momentum $\mathscr{P}^w_1$ carried by the waves, only a change in the vertical group velocity. The wave group coming from below is accelerated, the wave group coming from above decelerated. It is this asymmetric behaviour which causes a net momentum flux.

Assuming that the mean flow-induced changes are small, i.e. $|\Delta k_3| \ll k_3^0$, we find the momentum flux becomes

$$\mathscr{F}_{13} = -2\mathscr{N} k_1^2 \tau_R (\partial v_3^0/\partial k_3)\, \partial \bar{u}_1/\partial x_3, \tag{3.6}$$

or

$$F_{13} = -\int d^3k\, n(\mathbf{k})\, k_1^2 \tau_R (\partial v_3^0/\partial k_3)\, \partial \bar{u}_1/\partial x_3 \tag{3.7}$$

if we add up all the corresponding wave groups. The flux is proportional to the gradient of the mean flow. The factor of proportionality defines a wave-induced vertical viscosity coefficient

$$\nu_v = \int d^3k\, n(\mathbf{k})\, k_1^2 \tau_R\, \partial v_3^0/\partial k_3 = \int d^3k (\omega_0^2 - f^2)\, \omega_0^{-1} k_3 k_1^2 k^{-2} \partial(\tau_R n(\mathbf{k}))/\partial k_3, \tag{3.8}$$

where we have integrated by parts and substituted $v_3^0 = -(\omega_0^2 - f^2)\, k_3/\omega_0 k^2$. The basic structure of ν_v will be retained by our following, less heuristic, analysis.

4. Perturbation expansion

In order to solve the two basic coupled equations of the problem, the radiation-balance equation and the mean-flow equations, we treat the cross-coupling as a small perturbation. As the basic unperturbed state we consider an internal-wave

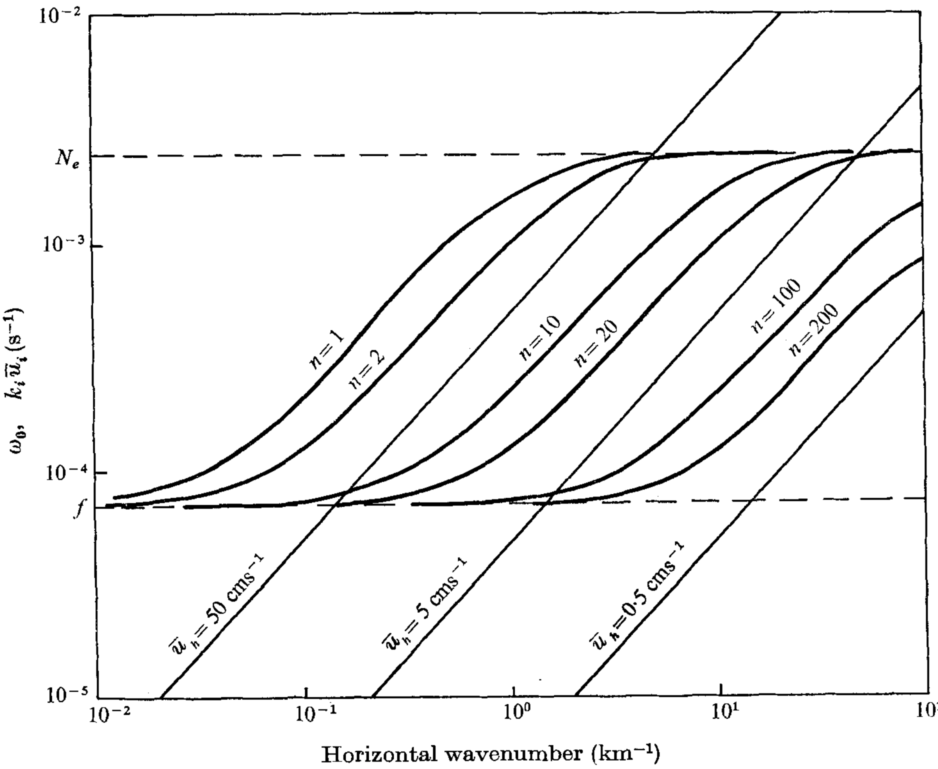

FIGURE 1. Intrinsic frequencies ω_0 for various values of the equivalent mode number n and Doppler shift $k_i \bar{u}_i$ for various values of the horizontal current velocity $\bar{u}_h$ plotted against horizontal wavenumber. The equivalent mode number is related to the vertical wavenumber by $|k_3| = n\pi N_e/N_0 b$, where $N_0 = 5{\cdot}2 \times 10^{-3}\,\text{s}^{-1}$ and $b = 1{\cdot}3\,\text{km}$ (Garrett & Munk 1972). The external parameters have been set at $N_e = 2{\cdot}6 \times 10^{-3}\,\text{s}^{-1}$ and $f = 7 \times 10^{-5}\,\text{s}^{-1}$.

field in a horizontally homogeneous, stationary, motionless and stratified ocean. In order that this unperturbed internal-wave field does not induce any mean motion it may only exert vertical forces which can be balanced by pressure forces. In the presence of a mean flow this basic state becomes modified. The mean flow affects the dispersion relation $\omega = \Omega_0 + \delta\Omega$ where Ω_0 is the unperturbed eigenfrequency in the absence of the mean flow and $\delta\Omega = k_i \bar{u}_i$ the Doppler shift induced by the mean flow. If most of the internal wave energy is confined to the first twenty modes (Garrett & Munk 1972), then, as figure 1 shows, the Doppler shift is small compared with the unperturbed frequency so long as the mean current velocity does not exceed $\bar{u}_h \approx 10$ cm/s. For regions with moderate mean currents we thus have $\gamma = |\delta\Omega|/\Omega_0 < 1$. The parameter γ, that is the ratio between the mean current and the phase velocity, will represent the formal expansion parameter for solving the radiation-balance equation. This expansion is not valid in regions with strong mean currents.

The Doppler shift induces perturbations in the group velocity and in the rate of refraction. The radiation-balance equation may then be written

$$(\mathscr{L}_0+\delta\mathscr{L})\,n = S[n], \tag{4.1}$$

where

$$\mathscr{L}_0 = \partial/\partial t+v_i^0\,\partial/\partial x_i+r_i^0\,\partial/\partial k_i \tag{4.2}$$

is the unperturbed Liouville operator and

$$\delta\mathscr{L} = \delta v_i\frac{\partial}{\partial x_i}+\delta r_i\frac{\partial}{\partial k_i} = \bar{u}_i\frac{\partial}{\partial x_i}-k_j\frac{\partial\bar{u}_j}{\partial x_i}\frac{\partial}{\partial k_i} \tag{4.3}$$

is the perturbation induced by the mean flow. The dominant term of $\mathscr{L}_0$ is the vertical group-velocity term $v_3^0\,\partial/\partial x_3$; the dominant term of $\delta\mathscr{L}$ is the refraction term $k_\alpha(\partial\bar{u}_\alpha/\partial x_3)\,\partial/\partial k_3$. Again we find $\delta\mathscr{L}/\mathscr{L}_0 = O(\gamma)$ if we restrict ourselves to moderate mean currents. More specifically we find from the condition

$$\left|k_\alpha\frac{\partial\bar{u}_\alpha}{\partial x_3}\frac{\partial}{\partial k_3}\right| \Big/ \left|v_3^0\frac{\partial}{\partial x_3}\right| < 1 \tag{4.4}$$

that our analysis is restricted to vertical shears

$$\left(\frac{\partial\bar{u}_\alpha}{\partial x_3}\frac{\partial\bar{u}_\alpha}{\partial x_3}\right)^{\frac{1}{2}} < \frac{v_v^0\,\Delta k_v}{k_h\,\Delta x_v} \approx 10\,\mathrm{cm\,s^{-1}/km}. \tag{4.5}$$

For this estimate we have taken the characteristic values $v_v^0 = 0{\cdot}9\,\mathrm{cm/s}$, $k_h = 1{\cdot}2\,\mathrm{km^{-1}}$, $\Delta k_v = n^{-1}\partial n/\partial k_3 = 12\,\mathrm{km^{-1}}$, and $\Delta x_v = n^{-1}\partial n/\partial x_3 = 1\,\mathrm{km}$ (cf. table 1). In order to solve the radiation-balance equation we expand the action-density spectrum n in powers of γ,

$$n = n^{(0)}+n^{(1)}+n^{(2)}+\dots. \tag{4.6}$$

Similarly we expand the source function

$$S[n] = S[n^{(0)}]+\delta S/\delta n\,[n^{(1)}+n^{(2)}+\dots], \tag{4.7}$$

neglecting for simplicity higher-order derivatives. Here $\delta S/\delta n$ denotes the functional derivative. Substituting these expansions into the radiation-balance equation we find at zeroth order

$$\mathscr{L}_0\,n^{(0)} = S[n^{(0)}], \tag{4.8}$$

which determines the unperturbed action-density spectrum $n^{(0)}$. Consistently we may assume that $n^{(0)}$ is stationary and horizontally homogeneous, i.e. $n^{(0)} = n^{(0)}(\mathbf{k},\ x_3)$. For such spectra the horizontal divergences of the wave-induced fluxes vanish: $\partial F_{i\alpha}^{(0)}/\partial x_\alpha = 0$, $\partial M_\alpha^{(0)}/\partial x_\alpha = 0$. So that the vertical divergences do not induce any mean motion we must have $\partial F_{\alpha 3}^{(0)}/\partial x_3 = 0$. This requirement can be met by an internal-wave field which is horizontally isotropic in wavenumber space. Hence, as a more restrictive condition, we assume $n^{(0)}$ to be independent of the direction of the horizontal wavenumber. There are no such requirements for the component $\partial F_{33}^{(0)}/\partial x_3$ since it can be balanced by the hydrostatic pressure. The unperturbed action-density spectrum may be envisaged as the spectrum proposed by Garrett & Munk (1972, 1975).

The first-order equation

$$\mathscr{L}_0 n^{(1)} + \delta\mathscr{L} n^{(0)} = \delta S/\delta n\,[n^{(1)}] \tag{4.9}$$

is formally solved by

$$n^{(1)} = -D^{-1}[Q], \tag{4.10}$$

where

$$D = \mathscr{L}_0 - \delta S/\delta n, \tag{4.11}$$

and

$$Q = \delta\mathscr{L} n^{(0)} = \bar{u}_3 \frac{\partial n^{(0)}}{\partial x_3} - k_j \frac{\partial n^{(0)}}{\partial k_i}\frac{\partial \bar{u}_j}{\partial x_i} = -k_\alpha \frac{\partial n^{(0)}}{\partial k_i}\frac{\partial \bar{u}_\alpha}{\partial x_i}. \tag{4.12}$$

Here we have used $\bar{u}_3 = 0$. The interaction causes a first-order modulation $n^{(1)}$ of the internal-wave field which is proportional to the spatial gradients of the mean flow.

The second-order equation

$$\mathscr{L}_0 n^{(2)} + \delta\mathscr{L} n^{(1)} = \partial S/\delta n\,[n^{(2)}] \tag{4.13}$$

becomes, when the first-order solution is substituted,

$$\mathscr{L}_0 n^{(2)} = \delta S/\delta n\,[n^{(2)}] + \delta\mathscr{L} D^{-1} \delta\mathscr{L} n^{(0)}. \tag{4.14}$$

When this equation is averaged over space and time and the different orders added up, the radiation-balance equation for the averaged internal-wave field takes the form

$$\mathscr{L}_0 \bar{n} = S[\bar{n}] + \overline{\delta\mathscr{L} D^{-1} \delta\mathscr{L} \bar{n}}. \tag{4.15}$$

Here the overbar denotes a space–time average over scales larger than the scales of the mean flow. Internal waves propagating in a random mean flow undergo secular changes described by the source term $\overline{\delta\mathscr{L} D^{-1} \delta\mathscr{L} \bar{n}}$. This process is analogous to the Fermi heating of electrons in a random time-dependent magnetic field, which is described by a Fokker–Planck equation, i.e. by a diffusion equation in velocity space. Similarly we find a diffusion equation in wavenumber space, that is

$$\overline{\delta\mathscr{L} D^{-1} \delta\mathscr{L}}\, \bar{n} = \frac{\partial}{\partial k_i} D_{ij} \frac{\partial}{\partial k_j} \bar{n}, \tag{4.16}$$

with

$$D_{ij} = k_\alpha \overline{\frac{\partial \bar{u}_\alpha}{\partial x_i} D^{-1} \frac{\partial \bar{u}_\beta}{\partial x_j}}\, k_\beta, \tag{4.17}$$

if $\delta\mathscr{L}$ is substituted from (4.12).

The effect of the modulation $n^{(1)}$ on the mean flow is determined by substituting $n^{(1)}$ into (2.41) for the wave-induced fluxes. These become

$$\left.\begin{aligned} F_{ij}^{(1)} &= \int d^3k f_{ij}\, n^{(1)} = \int d^3k f_{ij}\, D^{-1}\left[k_\alpha \frac{\partial}{\partial k_m} n^{(0)} \frac{\partial}{\partial x_m} \bar{u}_\alpha\right], \\ M_\beta^{(1)} &= \int d^3k m_\beta\, n^{(1)} = \int d^3k m_\beta\, D^{-1}\left[k_\alpha \frac{\partial}{\partial k_m} n^{(0)} \frac{\partial}{\partial x_m} \bar{u}_\alpha\right], \end{aligned}\right\} \tag{4.18}$$

and are proportional to the spatial gradients of the mean current velocity. The factors of proportionality define wave-induced diffusion coefficients. In the following sections we shall explicitly determine the modulation $n^{(1)}$ and the wave-induced diffusion coefficients. The evaluation of the Fokker–Planck term $\overline{\delta \mathscr{L} D^{-1} \delta \mathscr{L} n}$ and its implications for the energy balance of internal waves has been discussed in Müller & Olbers (1975).

5. Modulation of the internal wave field

To determine the mean-flow-induced modulation $n^{(1)}$, we must invert the operator $D = \mathscr{L}_0 - \delta S/\delta n$. Although the functional form of most processes which contribute to the source function S is known (Müller & Olbers 1975), the inversion of the functional derivative $\delta S/\delta n$ fails on account of the mathematical complexities. Nevertheless, the physical significance of $\delta S/\delta n$ is obvious.

Relaxation time. The source function S can be decomposed into three major components:

$$S = S_{\text{in}} + S_{\text{tr}} + S_{\text{diss}}, \tag{5.1}$$

representing the input of action into the wave field by external forces, the transfer (redistribution) of action within the wave field, and the dissipation of wave action respectively. The source term S_{in} is generally independent of the action-density spectrum. The irreversible part of the source terms S_{tr} and S_{diss} may be approximated by

$$S_{\text{tr}} + S_{\text{diss}} = -\tau_R^{-1} n, \tag{5.2}$$

where τ_R is the characteristic time scale of the transfer and dissipation processes. The radiation-balance equation then takes the form

$$\mathscr{L} n = S_{\text{in}} - \tau_R^{-1} n. \tag{5.3}$$

Here τ_R determines the equilibrium state of the wave field and the relaxation time, i.e. the characteristic time it takes for a disturbed internal-wave field to return to its equilibrium state. Having this simple structure in mind we approximate the functional derivative $\delta S/\delta n$ by the negative inverse of the relaxation time:

$$\delta S/\delta n = -\tau_R^{-1}(\mathbf{k}, x_3). \tag{5.4}$$

Here we have assumed that the relaxation time depends on the same variables as $n^{(0)}$.

Estimates of τ_R are given by Olbers (1976). Dividing the energy of the internal-wave field by the energy flux arising from resonant wave–wave interaction, he found as a characteristic value for the main thermocline $\tau_R = O(200\,\text{h})$ and a depth dependence $\tau_R \propto N_e^{-1}$. However, this value characterizes the relaxation time of the overall energy level. For our analysis the characteristic decay times of asymmetries and anisotropies will be of relevance. These can be expected to be smaller. Scattering at inhomogeneities of the density stratification (Mysak & Howe 1976) might provide an effective mechanism for the attenuation of asymmetries. It is not, however, clear from observations to what extent such inhomogeneities are due to persistent layers or to internal waves. There is also some empirical evidence for a smaller relaxation time (Frankignoul 1974, 1976). Nevertheless, Olbers's value may serve as an upper limit. There also exists a lower limit. Since we do not observe internal waves in the ocean the time scales of the dynamical processes that change the state of the wave field must be larger than the wave periods. Hence the mean value τ_R^0 of the relaxation time seems to be established to within one order of magnitude. As yet, however, there exists no definite information on the wavenumber dependence.

General solution. The operator $\delta S/\delta n$ having been approximated by the function $-\tau_R^{-1}(\mathbf{k}, x_3)$, the operator D can be inverted by integration along the characteristics. The result is

$$D^{-1}[Q] = \int_{-\infty}^{t} dt' Q(\mathbf{k}^0(t'), \mathbf{x}^0(t'), t') \exp\left\{-\int_{t'}^{t} dt'' \tau_R^{-1}(\mathbf{k}^0(t''), x_3^0(t''))\right\}. \quad (5.5)$$

Here $\mathbf{x}^0(t')$ and $\mathbf{k}^0(t')$ denote the unperturbed wave-group trajectories defined by

$$dx_i^0/dt = v_i^0, \quad x_i^0(t) = x_i; \quad dk_i^0/dt = r_i^0, \quad k_i^0(t) = k_i. \quad (5.6)$$

The wave field responds to all the forcing on its past trajectories. The modulation $n^{(1)}$ is a non-local functional of the forcing term $Q = -k_\alpha(\partial n^{(0)}/\partial k_i)\,\partial\bar{u}_\alpha/\partial x_i$, non-local in time, physical space and wavenumber space. However, owing to relaxation processes the wave field only retraces the forcing for a relaxation time, that is, the lower integration limit $-\infty$ may be replaced by $t-\tau_R$.

Local limit. For each of the independent variables of Q we introduce a propagation time τ_p. Each of these is defined as the time it takes for a wave group to change its position or wavenumber by an amount equal to one of the characteristic space or wavenumber scales of Q. The wavenumber scale of Q is given by the bandwidth of $n^{(0)}$. The propagation times represent the time scales of the unperturbed Liouville operator. In table 1 we have listed the characteristic length scales $\Delta\mathbf{x}$ and the characteristic wavenumber scales $\Delta\mathbf{k}$ of Q. Next to these, in column 2, we have listed mean values of the group velocity $\mathbf{v}^0$ and the rate of refraction $\mathbf{r}^0$. Column 3 lists characteristic values of the corresponding propagation times estimated by $\Delta x/v^0$ and $\Delta k/r^0$.

If the relaxation time τ_R is smaller than the propagation time τ_p the operator D^{-1} becomes local in the corresponding variable. Having an upper limit $\tau_R = O(200\,\mathrm{h})$, table 1 suggests that D^{-1} is local in the horizontal space co-ordinate and in the horizontal wavenumber. Additionally D^{-1} is local in the time co-ordinate since $\tau_R \ll T \approx 50$ days, T being the characteristic time scale of the mean flow. The operator D^{-1} may thus be approximated by

$$D^{-1}[Q] = \int_{-\infty}^{t} dt' Q(k_\alpha, k_3^0(t'), x_\alpha, x_3^0(t'), t) \exp\left\{-\int_{t'}^{t} dt'' \tau_R^{-1}(k_\alpha, k_3^0(t''), x_3^0(t''))\right\}. \quad (5.7)$$

The characteristics and the operator D^{-1} are now horizontally isotropic, i.e. independent of the direction of the horizontal wavenumber. This will simplify our algebra considerably.

If D^{-1} is local in all variables (i.e. $|\mathscr{L}_0| \ll \tau_R^{-1}$) we find

$$D^{-1}[Q] = \tau_R Q. \quad (5.8)$$

The effect of a possible non-locality is more fully discussed in Müller (1974). Here we only estimate its qualitative effect.

	Characteristic scale of Q	Group velocity or rate of refraction	Propagation time
Horizontal space co-ordinate	$\Delta x_h = 100\,\text{km}$	$\langle v_h^0 \rangle = 10\,\text{cm/s}$	280 h
Vertical space co-ordinate	$\Delta x_v = 1\,\text{km}$	$\langle v_v^0 \rangle = 0{\cdot}9\,\text{cm/s}$	30 h
Horizontal wavenumber	$\langle \Delta k_h \rangle = 0{\cdot}8\,\text{km}^{-1}$	$\langle r_h^0 \rangle = 0$	∞
Vertical wave-number	$\langle \Delta k_v \rangle = 12\,\text{km}^{-1}$	$\langle r_v^0 \rangle = 0{\cdot}25\,\text{km}^{-1}\,\text{h}^{-1}$	50 h

TABLE 1. Characteristic propagation times. The mean values $\langle \ldots \rangle$ have been calculated from the Garrett & Munk (1972) spectrum but only averaged from the 10th to the 20th mode.

The effect of non-locality. For simplicity we assume that N_e, $n^{(0)}$ and τ_R are independent of depth. The first-order equation (4.9) can now be solved by taking a Fourier transform with respect to the time and space variables since the wavenumber $\mathbf{k}$ enters the equation as a parameter only. For the Fourier components $n^{(1)}(\mathbf{k};\, \mathbf{K},\, \Omega)$ and $Q(\mathbf{k};\, \mathbf{K},\, \Omega) = -k_\alpha\,(\partial n^{(0)}/\partial k_i)\, iK_i\, \overline{u}_\alpha(\mathbf{K}, \Omega)$, (4.9) takes the form

$$(i\Omega + iK_i v_i^0 + \tau_R^{-1})\, n^{(1)}(\mathbf{k}; \mathbf{K}, \Omega) = -Q(\mathbf{k}; \mathbf{K}, \Omega), \tag{5.9}$$

and is solved by

$$n^{(1)}(\mathbf{k}; \mathbf{K}, \Omega) = -Q(\mathbf{k}; \mathbf{K}, \Omega)\,[\tau_R^{-1} - i(\Omega + K_i v_i^0)]/[\tau_R^{-2} + (\Omega + K_i v_i^0)^2]. \tag{5.10}$$

Relaxation effects induce the part which is in phase with the forcing $Q(\mathbf{k}; \mathbf{K}, \Omega)$. The energy exchange between the mean flow and the wave field is determined by $F_{\alpha j}^{(1)}\, \partial \overline{u}_\alpha/\partial x_j = \int d^3k f_{\alpha j}\, n^{(1)}\, \partial \overline{u}_\alpha/\partial x_j$. Only the in-phase part of $n^{(1)}$ contributes to its mean value. A secular energy transfer between the mean flow (forcing field) and the internal-wave field (forced field) occurs only if the forced field is damped or if it is forced in resonance. The resonance condition $\Omega + K_i v_i^0 = 0$ is, however, difficult to satisfy for typical deep ocean conditions. The order of magnitude of the in-phase part of $n^{(1)}$ is given by $n^{(1)} = D_{\text{eff}}^{-1}\, Q$, where

$$D_{\text{eff}}^{-1} = \tau_R/[1 + (\tau_R/\tau_p)^2] \approx 10\,\text{h}, \tag{5.11}$$

if we take $(\Omega + K_i v_i^0)^{-1} = \tau_p^{\min} \approx 30\,\text{h}$ and $\tau_R = 80\,\text{h}$. The specific value of D_{eff}^{-1} does not depend sensitively on the as yet unknown relaxation time τ_R. Within the possible interval $20\,\text{h} \leqslant \tau_R \leqslant 200\,\text{h}$ it changes by less than a factor of 3.

Correlation between internal-wave cross-spectra and mean-current gradients. The interaction with the mean flow causes a modulation of the internal-wave field which is both anisotropic and asymmetric in wavenumber space. These anisotropies and asymmetries are also apparent in the projections provided by the various measurement techniques.

Cross-spectra obtained from moored current meters and temperature sensors are defined by

$$\Gamma_{ij}(\omega) = C_{ij}(\omega) - iQ_{ij}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau \overline{u_i'(t)\,u_j'(t+\tau)}\, e^{-i\omega\tau}. \tag{5.12}$$

The vertical velocity is usually inferred from temperature series and the mean temperature gradient. If one calculates the covariance function $\overline{u_i'(t)\,u_j'(t+\tau)}$ with the help of (2.39), the cross-spectral matrix becomes a weighted projection of the action-density spectrum onto the frequency axis

$$\Gamma_{ij}(\omega) = \int d^3k \tfrac{1}{2} n(\mathbf{k})\, \omega \{U_i^{(0)} U_j^{(0)*} \delta(\omega - \Omega_0(\mathbf{k})) + U_i^{(0)*} U_j^{(0)} \delta(\omega + \Omega_0(\mathbf{k}))\}. \tag{5.13}$$

Here we have neglected the small Doppler shift term in the dispersion relation. In our case the action-density spectrum is given by $n = n^{(0)} + D^{-1}[k_\alpha(\partial n^{(0)}/\partial k_i) \times \partial \overline{u}_\alpha/\partial x_i]$. Hence we expect to find correlations between the cross-spectral matrix and the current gradients of the mean flow. As shown in the appendix we specifically expect correlations between

(i)	$C_{11}(\omega) - C_{22}(\omega)$,		$\partial\overline{u}_1/\partial x_1 - \partial\overline{u}_2/\partial x_2$,	(5.14*a*)
(ii)	$C_{12}(\omega)$,		$\partial\overline{u}_2/\partial x_1 + \partial\overline{u}_1/\partial x_2$,	(5.14*b*)
(iii)	$C_{13}(\omega)$,	$Q_{23}(\omega)$,	$\partial\overline{u}_1/\partial x_3$,	(5.14*c*)
(iv)	$C_{23}(\omega)$,	$Q_{13}(\omega)$,	$\partial\overline{u}_2/\partial x_3$.	(5.14*d*)

All other co- and quadrature spectra, especially $C_{11} + C_{22}$, should not be correlated with the mean current gradients. A correlation with the local gradients is expected if relaxation effects dominate propagation effects, i.e. $\tau_R < \tau_p$. Furthermore, determination of the regression lines provides information on the relaxation time.

The correlations suggested by our analysis are more specific than those suggested by phenomenological theories. There one closes the equations of motion by

$$F_{\alpha\beta} = -\nu_h^{\text{phen}}\, \partial\overline{u}_\alpha/\partial x_\beta, \quad F_{\alpha 3} = -\nu_v^{\text{phen}}\, \partial\overline{u}_\alpha/\partial x_3, \tag{5.15}$$

yielding only integral relations such as

$$\overline{u_1'u_1' - u_2'u_2'} = 2\int_f^{N_0} d\omega (C_{11}(\omega) - C_{22}(\omega)) = -\nu_h^{\text{phen}}\left(\frac{\partial\overline{u}_1}{\partial x_1} - \frac{\partial\overline{u}_2}{\partial x_2}\right). \tag{5.16}$$

The empirical verification of such integral relations is difficult since the integrated cross-spectra are dominated by inertial oscillations which, by any reasonable theory, do not effectively contribute to the wave-induced momentum flux. For testing the relationships (i) and (ii), Frankignoul (1974, 1976) calculated time series of $\partial\overline{u}_\beta/\partial x_\alpha$ and $C_{\alpha\beta}(\omega)$ from MODE data. He indeed found the suggested correlations and calculated a relaxation time $\tau_R = O(50\,\text{h})$ for the internal-wave continuum. His studies suggest that the local limit might be applicable.

There exist similar relationships for other measurement techniques. For cross-spectra obtained from dropped instruments (e.g. Sanford 1975) we expect to find correlations between

$$\left.\begin{aligned} C_{11}(k_3)-C_{22}(k_3), &\quad \partial\bar{u}_1/\partial x_1-\partial\bar{u}_2/\partial x_2, \\ C_{12}(k_3), &\quad \partial\bar{u}_2/\partial x_1+\partial\bar{u}_1/\partial x_2, \\ C_{\alpha 3}(k_3), &\quad \partial\bar{u}_\alpha/\partial x_3, \end{aligned}\right\} \tag{5.17}$$

if we assume $n^{(0)}$ to be vertically symmetric in wavenumber space. The correlations for towed spectra (e.g. Katz 1975) look more complicated.

6. Wave-induced diffusion coefficients

Diffusion operators. The reaction of the modulation $n^{(1)}$ on the mean flow is determined by substituting $n^{(1)}$ into (2.41) for the wave-induced fluxes. We obtain

$$F_{ij}^{(1)} = -N_{ij\beta m}[\partial\bar{u}_\beta/\partial x_m], \quad M_\alpha^{(1)} = -fK_{\alpha\beta m}[\partial\bar{u}_\beta/\partial x_m], \tag{6.1}$$

where the wave-induced viscosity and diffusivity tensors are

$$\left.\begin{aligned} N_{ij\beta m}[\ldots] &= -\int d^3k f_{ij}\, D^{-1}[k_\beta\, \partial n^{(0)}/\partial k_m \ldots], \\ K_{\alpha\beta m}[\ldots] &= -f^{-1}\int d^3k m_\alpha\, D^{-1}[k_\beta\, \partial n^{(0)}/\partial k_m \ldots]. \end{aligned}\right\} \tag{6.2}$$

The diffusion tensors depend on the unperturbed internal-wave field $n^{(0)}$ and on its relaxation time τ_R. They act as non-local operators on the mean flow. The structure of the diffusion terms is considerably simplified if we carry out the integration over the direction ϕ of the horizontal wavenumber using the fact that $n^{(0)}$ and D^{-1} are horizontally isotropic. The integrals involved are of the form

$$\int_0^{2\pi} d\phi \sin^n\phi \cos^m\phi,\ m+n \leqslant 4.$$

Referring to the appendix for the details we find that the equations of motion for the mean flow reduce to

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\bar{u}_\alpha-\epsilon_{\alpha\beta}f\bar{u}_\beta+\frac{\partial}{\partial x_\alpha}\pi_f=\frac{\partial}{\partial x_\beta}N_h\left[\frac{\partial}{\partial x_\beta}\bar{u}_\alpha\right]+\frac{\partial}{\partial x_3}N_v\left[\frac{\partial}{\partial x_3}\bar{u}_\alpha\right], \tag{6.3a}$$

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\bar{u}_3-\bar{b}+\frac{\partial}{\partial x_3}\pi_f=0, \tag{6.3b}$$

$$\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\bar{b}+N_e^2\bar{u}_3=f\frac{\partial}{\partial x_\alpha}K_h\left[\epsilon_{\alpha\beta}\frac{\partial}{\partial x_3}\bar{u}_\beta\right]. \tag{6.3c}$$

The horizontal and vertical viscosity and horizontal diffusivity operators are given by

$$N_h[\ldots] = -\tfrac{1}{8}\int d^3k\omega_0 k_3^2 k^{-2}(\omega_0^2-f^2)\,\omega_0^{-2}\, D^{-1}[k_\alpha\, \partial n^{(0)}/\partial k_\alpha\ldots], \tag{6.4a}$$

$$N_v[\ldots] = \tfrac{1}{2}\int d^3k\omega_0 k_3 k_\alpha k^{-2} D^{-1}[k_\alpha\, \partial n^{(0)}/\partial k_3\ldots], \tag{6.4b}$$

$$K_h[\ldots] = -\tfrac{1}{2}\int d^3k\omega_0 k_3 k_\alpha k^{-2} N_e^2\,\omega_0^{-2}\, D^{-1}[k_\alpha\, \partial n^{(0)}/\partial k_3\ldots]. \tag{6.4c}$$

The momentum balance has the familiar form: a diagonal viscosity tensor in the horizontal momentum balance and no diffusion term in the vertical momentum balance. Also, there is no vertical diffusion of buoyancy and other passive properties. The horizontal diffusion of buoyancy is due to the vertical gradient of the mean current. However, if we substitute the thermal-wind relations $f\epsilon_{\alpha\beta}\,\partial\bar{u}_\beta/\partial x_3 = \partial\bar{b}/\partial x_\alpha$ the usual form

$$f\frac{\partial}{\partial x_\alpha}K_h\left[\epsilon_{\alpha\beta}\frac{\partial}{\partial x_3}\bar{u}_\beta\right] = \frac{\partial}{\partial x_\alpha}K_h\left[\frac{\partial}{\partial x_\alpha}\bar{b}\right] \tag{6.5}$$

is obtained. For the potential-vorticity equation we find

$$\left(\frac{\partial}{\partial t}+\bar{u}_\alpha\frac{\partial}{\partial x_\alpha}\right)\left(\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}+\frac{\partial}{\partial x_3}\frac{f^2}{N_e^2}\frac{\partial}{\partial x_3}\right)\phi = \left(\frac{\partial}{\partial x_\alpha}D_h\frac{\partial}{\partial x_\alpha}+\frac{\partial}{\partial x_3}D_v\frac{\partial}{\partial x_3}\right)\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}\phi, \tag{6.6}$$

with

$$D_h = N_h, \quad D_v = N_v + f^2K_h/N_e^2. \tag{6.7}$$

The diffusion tensor has the usual diagonal form. Note that D_v closely resembles our heuristic estimate (3.8) if $D^{-1} = \tau_R$.

Diffusion coefficients. In the local limit the operator D^{-1} can be approximated by the relaxation time τ_R and the diffusion operators become numbers: the usual diffusion coefficients ν_h, ν_v and κ_h. Let us assume $\tau_R(\mathbf{k}) = \tau_R^0 =$ constant. Integrating (6.4) by parts and changing from the **k**-representation to the (k_1, k_2, ω_0, s)-representation $(s = \operatorname{sgn} k_3)$ we find

$$\left.\begin{matrix}\nu_h\\ \nu_v\\ \kappa_h\end{matrix}\right\} = \tau_R^0\int_f^{N_e} d\omega_0 \left\{\begin{matrix}W_1(\omega_0)\\ W_2(\omega_0)\\ W_3(\omega_0)\end{matrix}\right\}\sum_{s=\pm 1}\int dk_1\,dk_2\,\omega_0\,n(k_1,k_2,\omega_0,s), \tag{6.8}$$

with the weighting functions

$$\left.\begin{matrix}W_1\\ W_2\\ W_3\end{matrix}\right\} = \frac{1}{2}\frac{\omega_0^2-f^2}{\omega_0^2}\frac{1}{N_e^4}\left\{\begin{matrix}\tfrac{1}{4}(N_e^2-\omega_0^2)\,\omega_0^{-2}(3N_e^2\omega_0^2+N_e^2f^2-3\omega_0^4), & (6.9a)\\ 2N_e^2\omega_0^2-N_e^2f^2-3\omega_0^4, & (6.9b)\\ \omega_0^{-2}N_e^2(\omega_0^4-N_e^2f^2). & (6.9c)\end{matrix}\right.$$

Here we have neglected terms $O(f^2/N_e^2)$ for simplicity. The diffusion coefficients are weighted integrals of the unperturbed action-density spectrum. The weighting functions depend on the frequency only. Hence no information on the wavenumber dependence of $n^{(0)}$ is needed, only its projection onto the frequency axis

$$e(\omega_0) = \sum_s\int dk_1\,dk_2\,\omega_0\,n^{(0)}(k_1,k_2,\omega_0,s). \tag{6.10}$$

Figure 2(*a*) shows the weighting functions W_i $(i = 1, 2, 3)$ for $N_e/f = 37{\cdot}5$ $(N_e = 2{\cdot}6\times10^{-3}\,\mathrm{s}^{-1},\ f = 7\times10^{-5}\,\mathrm{s}^{-1})$. The weighting functions are zero in the limit $\omega_0\to f\ (k_{1,2}\to 0)$. Hence inertial oscillations do not contribute to the diffusion coefficients. The weighting function W_1 is also zero in the limit $\omega_0\to N_e\ (k_3\to 0)$. Hence buoyancy oscillations do not contribute to the horizontal diffusion of momentum. The weighting functions W_2 and W_3 are partly negative. The sign of the corresponding diffusion coefficients therefore depends on the shape of frequency spectrum $e(\omega_0)$. The level, shape and depth dependence of the frequency spectrum $e(\omega_0)$ are well established and can be fitted analytically by (Garrett & Munk 1972, 1975)

$$e(\omega_0) = Ef\omega_0^{-r}, \quad \text{for} \quad r = 2, \tag{6.11}$$

$$E(x_3) = E_0N_e(x_3)/N_0, \quad E_0 = 30\,\mathrm{cm}^2/\mathrm{s}^2, \quad N_0 = 5{\cdot}2\times10^{-3}\,\mathrm{s}^{-1},$$

if we do not consider the inertial peak.

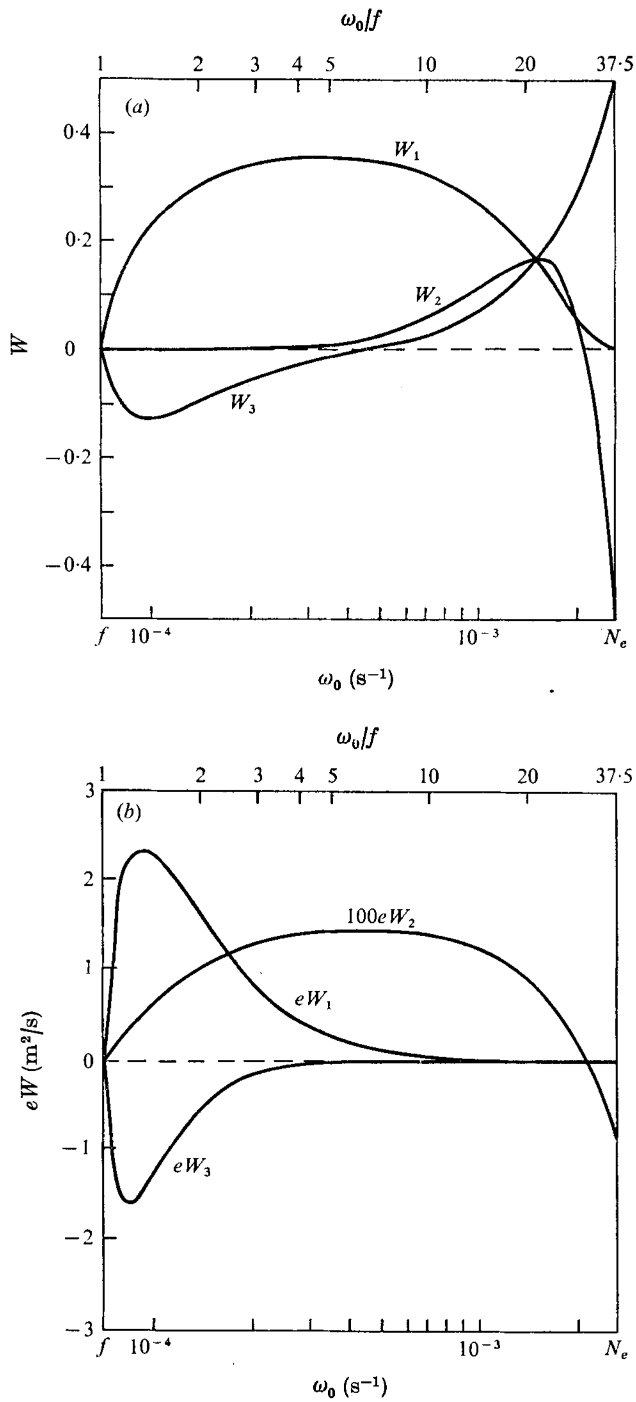

FIGURE 2. Frequency dependence of (*a*) the weighting functions W_i and (*b*) the integrands eW_i.

$e(\omega_0)$ $\tau_R(\mathbf{k})$	$2\pi^{-1}E\omega_0^{-1}f(\omega_0^2-f^2)^{-\frac{1}{2}}$ τ_R^0	$E\frac{3}{2}f^{\frac{3}{2}}\omega_0^{-\frac{5}{2}}$ τ_R^0	$E\frac{1}{2}f^{\frac{1}{2}}\omega_0^{-\frac{3}{2}}$ τ_R^0	$Ef\omega_0^{-2}$ $\tau_R^0(Nf)^{\frac{1}{2}}\omega_0^{-1}$
ν_h	$0{\cdot}20\,E\tau_R^0$	$0{\cdot}23\,E\tau_R^0$	$0{\cdot}31\,E\tau_R^0$	$0{\cdot}079\,E\tau_R^0(N/f)^{\frac{1}{2}}$ $0{\cdot}48\,E\tau_R^0$
ν_v	$0{\cdot}32\,E\tau_R^0 f/N_e$	$2{\cdot}1\,E\tau_R^0(f/N_e)^{\frac{3}{2}}$ $0{\cdot}34\,E\tau_R^0 f/N_e$	$0{\cdot}12\,E\tau_R^0(f/N_e)^{\frac{1}{2}}$ $0{\cdot}73\,E\tau_R^0 f/N_e$	$\frac{1}{2}E\tau_R^0(f/N)^{\frac{3}{2}}[\ln(N_e/f)-1{\cdot}51]$ $0{\cdot}17\,E\tau_R^0 f/N_e$
κ_h	$-0{\cdot}063\,E\tau_R^0$	$-0{\cdot}048\,E\tau_R^0$	$-0{\cdot}044\,E\tau_R^0$	$-0{\cdot}125\,E\tau_R^0(N_e/f)^{\frac{1}{2}}$ $-0{\cdot}75\,E\tau_R^0$

TABLE 2. Diffusion coefficients for various forms of the frequency spectrum $e(\omega_0)$ and for a relaxation time $\tau_R(\mathbf{k})$ proportional to the period. When the functional dependence on N_e/f differs from the one in (6.12) the lower set of values was calculated from $N_e/f = 37{\cdot}5$.

Using this analytical fit the frequency integration in (6.8) can be carried out analytically. The complete integrands $W_i(\omega_0)\,e(\omega_0)$ for this case are shown in figure 2 (*b*). The result of the integration is

$$\left.\begin{matrix}\nu_h\\ \nu_v\\ \kappa_h\end{matrix}\right\} = E\tau_R^0\left\{\begin{matrix}0{\cdot}25,\\ 0{\cdot}50\,f/N_e,\\ -0{\cdot}067.\end{matrix}\right\} \tag{6.12}$$

In order to investigate the sensitivity of this result we have listed in table 2 the corresponding results for modified frequency spectra and for a frequency-dependent relaxation time. In column 1 we have included an inertial peak (Garrett & Munk 1972, 1975). The functional dependence on E, τ_R^0, and N_e/f is the same. The numerical factors are changed by less than a factor of 2. The result is also not very sensitive to the exponent r in the power law (6.11). Varying r within the interval $1{\cdot}5 \leqslant r \leqslant 2{\cdot}5$ changes the result by less than a factor of 3 although the dependence of ν_v on N_e/f is slightly modified (columns 2 and 3). Finally column 4 lists the results for a relaxation time which is proportional to the period. Such a behaviour of the relaxation time can be expected on intuitive grounds. It is also consistent with Olbers's (1974) findings that, at least for high frequencies, monochromatic beams of internal waves are damped by resonant wave–wave interaction according to $\tau_R \propto \omega_0^{-n}$ $(0 < n < 1)$. Again no dramatic changes occur.

We have not yet taken into account that our WKBJ approximation of the internal-wave field is only applicable to that part of the wave energy which is in high mode numbers. Denoting this fraction by q we have to replace the total energy density E by qE in (6.12). Furthermore, if the diffusion process is non-local its qualitative effect may be estimated by replacing the relaxation time τ_R^0 by a characteristic value of D_{eff}^{-1}.

The viscosity coefficients ν_h and ν_v are positive. The horizontal diffusivity is negative. This conflicts somewhat with our physical intuition. However, since we are restricted to a quasi-geostrophic mean flow the diffusion coefficients $d_h = \nu_h$ and $d_v = \nu_v + \kappa_h f^2/N_e^2$ in the potential-vorticity equation are relevant.

Both of these are positive. It should, however, be noted that the sign of the vertical diffusion coefficient d_v depends on the shape of the frequency spectrum $e(\omega_0)$. There are positive and negative contributions to it. On our level of analysis there is no intrinsic reason for the diffusion coefficient to be positive.

The Brunt–Väisälä frequency N_e, the energy density E and the relaxation time τ_R depend on depth and determine implicitly the depth dependence of the viscosity coefficients. For resonant wave–wave interaction Olbers (1976) estimated $\tau_R \propto N_e^{-1}$. This together with $E \propto N_e$ yields

$$\nu_h, \kappa_h \propto \text{constant}, \quad \nu_v \propto N_e^{-1} \tag{6.13}$$

in the local limit. However the local limit is less applicable in the deep water since τ_R increases with depth whereas $\tau_p \propto 1/v_v^0 \propto N_e$ decreases with depth. A more realistic estimate is presumably

$$\nu_h^{\text{eff}}, \kappa_h^{\text{eff}} \propto N_e^4, \quad \nu_v^{\text{eff}} \propto N_e^3, \tag{6.14}$$

based on $D_{\text{eff}}^{-1} \propto N_e^3$ for $\tau_p > \tau_R$. Also, our estimates (6.12) are not applicable to the seasonal thermocline where trapping of the internal wave field hinders free propagation.

Order of magnitude. The eddy coefficients can easily be evaluated, the two unknowns being τ_R^0 and q. Accepting the model of Garrett & Munk (1972) and assuming that our WKBJ approximation is sufficiently correct for mode numbers $n \geqslant 10$ implies $q = \frac{1}{2}$. Taking this value and taking from (5.11) $D_{\text{eff}}^{-1} = 10\,\text{h}$ yields for the main thermocline ($N_e = 2{\cdot}6 \times 10^{-3}\,\text{s}^{-1}, f = 7 \times 10^{-5}\,\text{s}^{-1}$)

$$\nu_h^{\text{eff}} = 7 \times 10^4\,\text{cm}^2/\text{s}, \quad \nu_v^{\text{eff}} = 4 \times 10^3\,\text{cm}^2/\text{s}, \quad \kappa_h^{\text{eff}} = -2 \times 10^4\,\text{cm}^2/\text{s}. \tag{6.15}$$

The value of the vertical viscosity is considerably larger than the usual $1\,\text{cm}^2/\text{s}$ used in models of the general circulation. This discrepancy can mainly be ascribed to the fact that wave groups that propagate almost freely transport momentum over much larger distances than turbulent eddies do.

What are the possible errors in these values? The absolute value of E and the bandwidth of the internal-wave field are trustworthy within a factor of 3. The value of D_{eff}^{-1} does not change by more than a factor of 3 within the relevant interval $20\,\text{h} \leqslant \tau_R \leqslant 200\,\text{h}$. Another factor of 3 arises from the variability described in table 2. The least defined quantity is the fraction q of the total internal-wave energy for which the WKBJ approximation is correct. As regards the validity of the WKBJ approximation in the vertical space co-ordinate our value of $q = \frac{1}{2}$ is conceivable. However, there exist also restrictions from the horizontal space co-ordinate, since the horizontal wavelength approaches infinity for nearly inertial oscillations. These can be disregarded by changing the lower integration limit in (6.8) from f to $2f$. This does not affect the vertical viscosity coefficient. The horizontal viscosity is reduced by a factor of $\frac{1}{2}$. The horizontal diffusivity is reduced by a factor of $\frac{1}{8}$, but this is insignificant because κ_h does not contribute effectively to the vertical diffusion of vorticity. This discussion suggests that an error or variability by a factor of 5 seems conceivable for the viscosity coefficients. The horizontal diffusivity may be more uncertain since its value depends more sensitively on the validity of the WKBJ approximation for nearly inertial oscillations.

Energy dissipation. The diffusion processes within the mean flow dissipate energy at a rate

$$\dot{E} = -\rho_0 \nu_h^{\mathrm{eff}} \frac{\partial \bar{u}_\beta}{\partial x_\alpha}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} - \rho_0 \nu_v^{\mathrm{eff}} \frac{\partial \bar{u}_\alpha}{\partial x_3}\frac{\partial \bar{u}_\alpha}{\partial x_3} - \rho_0 N_e^{-2} \kappa_h^{\mathrm{eff}} \frac{\partial \bar{b}}{\partial x_\alpha}\frac{\partial \bar{b}}{\partial x_\alpha}, \tag{6.16}$$

where

$$\dot{E}_h = -\rho_0 \nu_h^{\mathrm{eff}} \frac{\partial \bar{u}_\beta}{\partial x_\alpha}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} = -2\times 10^{-8}\,\mathrm{erg/cm^3\,s}, \tag{6.17a}$$

$$\dot{E}_v = -\rho_0 \nu_v^{\mathrm{eff}} \frac{\partial \bar{u}_\alpha}{\partial x_3}\frac{\partial \bar{u}_\alpha}{\partial x_3} = -10^{-5}\,\mathrm{erg/cm^3\,s}, \tag{6.17b}$$

$$\dot{E}_\kappa = -\rho_0 \kappa_h^{\mathrm{eff}} N_e^{-2} \frac{\partial \bar{b}}{\partial x_\alpha}\frac{\partial \bar{b}}{\partial x_\alpha} = 4\times 10^{-8}\,\mathrm{erg/cm^3\,s}, \tag{6.17c}$$

if we take

$$\left(\frac{\partial \bar{u}_\beta}{\partial x_\alpha}\frac{\partial \bar{u}_\beta}{\partial x_\alpha}\right)^{\frac{1}{2}} = 5\,\mathrm{cm\,s^{-1}}/100\,\mathrm{km}, \quad \left(\frac{\partial \bar{u}_\alpha}{\partial x_3}\frac{\partial \bar{u}_\alpha}{\partial x_3}\right)^{\frac{1}{2}} = 5\,\mathrm{cm\,s^{-1}}/1\,\mathrm{km},$$

$$\left(\frac{\partial \bar{b}}{\partial x_\alpha}\frac{\partial \bar{b}}{\partial x_\alpha}\right)^{\frac{1}{2}} = f\left(\frac{\partial \bar{u}_\alpha}{\partial x_3}\frac{\partial \bar{u}_\alpha}{\partial x_3}\right)^{\frac{1}{2}}.$$

The energy dissipation due to the vertical diffusion of momentum considerably exceeds the dissipation due to the horizontal diffusion of mass and momentum. The dissipation rate per unit surface area may be approximated by

$$\dot{\mathscr{E}} = \dot{E}_v \times 1\,\mathrm{km} = 1\,\mathrm{erg/cm^2\,s}. \tag{6.18}$$

The dissipated energy appears in the internal-wave field which causes the diffusion. The distribution of the dissipated energy among the different wave numbers is given by the Fokker–Planck term (4.16) and is discussed in Müller & Olbers (1975).

7. Comparison with existing observations and concepts

Although this paper primarily aims at the presentation of the theoretical analysis, the exceptionally high value of the vertical viscosity and its associated dissipation rate requires some comparison with existing observations and concepts. Direct measurements of eddy or wave-induced viscosity coefficients in the deep ocean are sparse. Such measurements require the simultaneous recording of the small-scale fluctuations and the large-scale gradients. Based on the correlations suggested by our analysis, Frankignoul (1974, 1976) estimated the horizontal viscosity $\nu_h(\omega)$ for various internal-wave frequency bands from MODE data. His estimates are consistent with our results.

General circulation. Typical values of the diffusion coefficients used in models of the general circulation are $\nu_h = 10^7\,\mathrm{cm^2/s}$, $\nu_v = 1\,\mathrm{cm^2/s}$, $\kappa_h = 10^6\,\mathrm{cm^2/s}$, $\kappa_v = 1\,\mathrm{cm^2/s}$ implying Prandtl numbers $P_h = \nu_h/\kappa_h = 10$ and $P_v = 1$. The vertical diffusion coefficients are representative for most three-dimensional models. The horizontal coefficients may vary by two orders of magnitude depending on the horizontal grid scale and the processes which are to be simulated. Our estimate $\nu_h^{\text{eff}} = O(10^5\,\text{cm}^2/\text{s})$ is much smaller, indicating that the internal wave field does not effectively contribute to the horizontal diffusion of momentum in the general circulation. The main contribution arises presumably from the interaction with meso-scale eddies. Our estimate $\nu_v^{\text{eff}} = O(10^3\,\text{cm}^2/\text{s})$, on the contrary, suggests a much more effective vertical diffusion of momentum. Its detailed consequences can however only be inferred when included in the numerical models. The characteristic features of the general circulation are not changed significantly. The Ekman number is still much smaller than unity, $Ek = \nu_v^{\text{eff}}/fL_v^2 = O(10^{-3})$. The thickness of the Ekman boundary layer, $d = (2\nu_v^{\text{eff}}/f)^{\frac{1}{2}} = O(100\,\text{m})$, is reasonable if our value is valid near the surface and the bottom.

Meso-scale eddies. The nonlinear time-dependent meso-scale motion in the open ocean has been simulated numerically by Bretherton & Karweit (1974) and Rhines (1973, 1975). Both neglect the interaction with the subgrid component of the motion. Also, most attempts to fit observed features of meso-scale motions have managed without eddy or wave-induced diffusion. A noteworthy exception is the analysis of current profiles observed in the Polygon experiment by Fomin (unpublished work, 1972). Balancing the observed ageostrophy by vertical friction he found a vertical eddy-viscosity coefficient which is inversely proportional to N_e which changes from 10–$10^2\,\text{cm}^2/\text{s}$ in the upper part of the ocean to 10^4–$10^5\,\text{cm}^2/\text{s}$ at great depth. Although these findings would support our estimates in the local limit it should be noted that the complex vertical structure observed in the Polygon experiment is not a typical feature of meso-scale motions.

The recognition of an effective vertical diffusion of momentum in the open ocean is presumably hindered by the fact that the vertical diffusion is masked or competes with other processes. Open-ocean eddies, for example, as observed in MODE, may well represent a balance between a continuous forcing (by baroclinic instability of the general circulation, by the atmosphere or by the Gulf Stream) and the dissipative interaction with internal waves. The effect of the vertical viscosity should be more obvious when we consider the decay of Gulf Stream rings, which are generated by discrete events.

Gulf Stream rings. Observed decay times (*e*-folding times) (Fuglister 1971; Barrett 1971; Saunders 1971; Parker 1971; Cheney & Richardson 1976) range from a few months up to a year, although the ring may be recognized considerably longer. Similar decay times are observed for rings formed by the Kuroshio (Kitano 1975). Various mechanisms have been made responsible for the decay of the rings: heat loss to the atmosphere for anticyclonic rings (Saunders 1971); lateral momentum and density diffusion (Molinari 1970); vertical friction (Saunders 1971). Without attempting to model the actual decay of a ring, here we only demonstrate that decay times based on our estimates of the viscosity coefficients are not inconsistent with the observed decay times. Estimates of the decay time based on the dissipation rates $\dot{E}$ or $\dot{\mathscr{E}}$ in (6.17) and (6.18) and the kinetic energy density of the ring are misleading. Friction causes dissipation of kinetic energy, but this dissipation is partly balanced by an internal conversion of available potential energy in order to maintain the geostrophic balance. Our estimate of the decay times will therefore be based on the linearized potential-vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}+\frac{\partial}{\partial x_3}\frac{f^2}{N_e^2}\frac{\partial}{\partial x_3}\right)\phi+\beta\frac{\partial}{\partial x_1}\phi=\left(\frac{\partial}{\partial x_\alpha}d_h\frac{\partial}{\partial x_\alpha}+\frac{\partial}{\partial x_3}d_v\frac{\partial}{\partial x_3}\right)\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}\phi, \quad (7.1\,a)$$

$$\frac{\partial}{\partial t}\left(N_e^2+g\frac{\partial}{\partial x_3}\right)\phi=0 \quad \text{at} \quad x_3=0, \quad (7.1\,b)$$

$$\partial^2\phi/\partial t\,\partial x_3=0 \quad \text{at} \quad x_3=-h_0, \quad (7.1\,c)$$

which describes the damping of planetary Rossby waves.

Assuming separable eigensolutions of the form

$$\phi(\mathbf{x},t)=\psi_n(x_3)\exp\{i(K_\alpha x_\alpha-\Omega t)\}, \quad n=0,1,2,\ldots,$$

we find that the vertical eigenfunctions $\psi_n(x_3)$ are determined by

$$\frac{\partial}{\partial x_3}\frac{f^2}{N_e^2}\frac{\partial}{\partial x_3}\psi_n+\frac{iK^2}{\Omega}\left(\frac{\partial}{\partial x_3}d_v\frac{\partial}{\partial x_3}-K^2d_h\right)\psi_n=-\lambda_n\psi_n, \quad (7.2\,a)$$

$$\left(N_e^2+g\frac{\partial}{\partial x_3}\right)\psi_n=0 \quad \text{at} \quad x_3=0, \quad (7.2\,b)$$

$$\frac{\partial\psi_n}{\partial x_3}=0 \quad \text{at} \quad x_3=-h_0, \quad (7.2\,c)$$

where the eigenvalue $$\lambda_n=-K^2-\beta K_1/\Omega, \quad (7.3)$$

and $K^2=K_\alpha K_\alpha$. The viscosity terms represent a small correction of the operator on the left-hand side of (7.2 a) as long as $d_v K^2/\Omega, d_h K^2 L_v^2/\Omega L_h^2 \ll f^2/N_e^2$. In this case the vertical eigenvalue problem can be solved by a perturbation expansion (Landau & Lifshitz 1962)

$$\psi_n=\psi_n^{(0)}+\psi_n^{(1)}+\ldots, \quad \lambda_n=\lambda_n^{(0)}+\lambda_n^{(1)}+\ldots. \quad (7.4)$$

The zero-order solution defines undamped Rossby waves with frequency $\Omega_n=-\beta K_1/[K^2+\lambda_n^{(0)}]$ with $(\lambda_n^{(0)})^{-\frac{1}{2}}$ being the Rossby radius of deformation. For the first-order correction of the eigenvalue we find

$$\lambda_n^{(1)}=-\frac{iK^2}{\Omega}\int_{-h_0}^{0}dx_3\,\psi_n^{(0)}\left(\frac{\partial}{\partial x_3}d_v\frac{\partial}{\partial x_3}-K^2d_h\right)\psi_n^{(0)}. \quad (7.5)$$

The eigenfrequency is now obtained from (7.3) as

$$\Omega_n=-\frac{\beta K_1}{K^2+\lambda_n^{(0)}}+i\frac{K^2}{K^2+\lambda_n^{(0)}}\int_{-h_0}^{0}dx_3\,\psi_n^{(0)}\left(\frac{\partial}{\partial x_3}d_v\frac{\partial}{\partial x_3}-K^2d_h\right)\psi_n^{(0)}. \quad (7.6)$$

The real part remains unchanged. The imaginary part defines the characteristic time scale of the diffusion terms in the potential-vorticity equation. For $N_e^2=$ constant, we have $\psi_n^{(0)}\sim\cos K_3^n(x_3+h_0)$ and $\lambda_n^{(0)}=(K_3^n)^2f^2/N_e^2$, where $K_3^0\approx N_e(gh_0)^{-\frac{1}{2}}$ and $K_3^n\approx n\pi/h_0$ for $n=1,2,\ldots$. Assuming additionally $d_{h,v}=d_{h,v}^{\text{eff}}=$ constant, the decay time for the energy is given by

$$T_n=\tfrac{1}{2}K^{-2}(K^2+\lambda_n^{(0)})\,(K^2d_h^{\text{eff}}+(K_3^n)^2d_v^{\text{eff}})^{-1}. \quad (7.7)$$

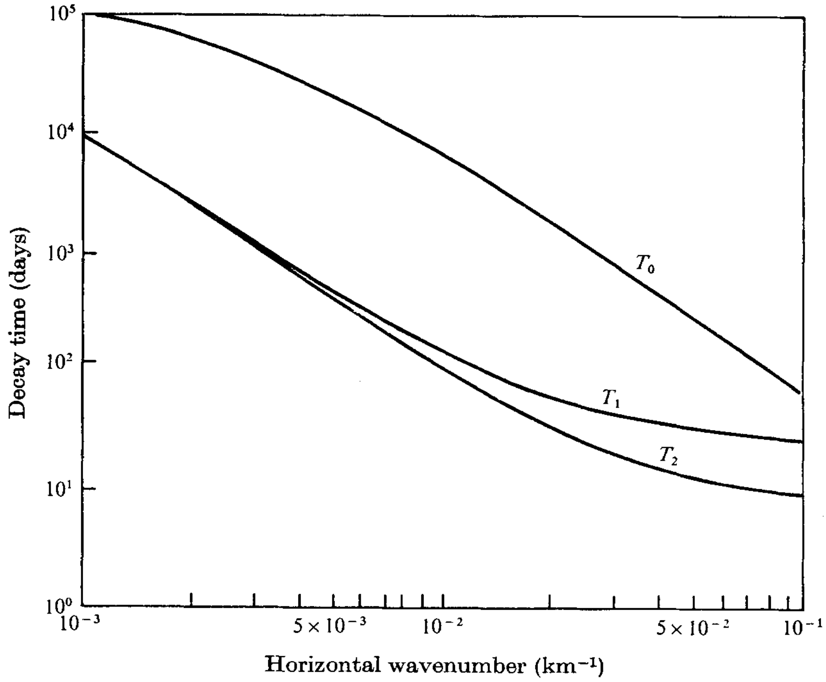

FIGURE 3. Decay time T_n versus horizontal wavenumber K for the barotropic, and first and second baroclinic mode. The external parameters have been set at $N_e = 2{\cdot}6 \times 10^{-3}\,\mathrm{s}^{-1}$, $= 7 \times 10^{-5}\,\mathrm{s}^{-1}$, $h_0 = 5\,\mathrm{km}$, $d_h^{\mathrm{eff}} = 7 \times 10^4\,\mathrm{cm}^2/\mathrm{s}$, $d_v^{\mathrm{eff}} = 4 \times 10^3\,\mathrm{cm}^2/\mathrm{s}$.

Figure 3 shows T_n as a function of K for the first 3 modes. For $K = O(2\pi/400\,\mathrm{km})$ typical values are a few months for the first baroclinic mode and a few years for the barotropic mode. For the actual modelling of the decay of a ring we must consider that our estimates of d_h^{eff} and d_v^{eff} do not apply to the seasonal thermocline. Vertical friction may however be responsible for the decay of ring energy in the main thermocline.

Energetics. The interaction of the meso-scale eddy field with the internal-wave field provides an effective dissipation mechanism for eddy energy. In Müller & Olbers (1975) an energy balance of the internal-wave field was proposed where this interaction also provides the main energy source of the internal-wave field. The balance suggests that the energy gained by the interaction with the mean flow is transferred down the spectrum by wave–wave interaction and is dissipated by wave breaking. Wave breaking partly dissipates energy into smaller-scale turbulence and partly increases the mean potential energy by mixing. Since wave–wave interaction conserves energy this energy balance may be written

$$\nu_v^{\mathrm{eff}} \frac{\partial \bar{u}_\alpha}{\partial x_3} \frac{\partial \bar{u}_\alpha}{\partial x_3} - \epsilon - \kappa_v N_e^2 = 0, \tag{7.8}$$

where ϵ is the dissipation rate to smaller-scale turbulence and κ_v the equivalent vertical diffusivity. From tank experiments Thorpe (1973) proposed $\epsilon/\kappa_v N_e^2 \approx 3$.

Taking this ratio we obtain

$$\kappa_v = \tfrac{1}{4}\nu_v^{\text{eff}} \frac{\partial \bar{u}_\alpha}{\partial x_3}\frac{\partial \bar{u}_\alpha}{\partial x_3} N_e^{-2} = \tfrac{1}{4}\nu_v^{\text{eff}} R_i^{-1} = 0{\cdot}4\,\text{cm}^2/\text{s}, \tag{7.9}$$

not inconsistent with recent estimates (Roether *et al.* 1970; Rooth & Östlund 1972). From (7.9) we find a Prandtl number

$$P_v = \nu_v^{\text{eff}}/\kappa_v = 4\,Ri \tag{7.10}$$

much larger than unity, since the diffusion of momentum and mass is caused by different processes, the diffusion of momentum being caused by wave propagation, the diffusion of mass by wave breaking.

Our dissipation rate for eddy energy is also comparable with the production of eddy energy if we assume that an appreciable part of the energy input into the general circulation by the wind stress, $\dot{\mathscr{E}}_T \approx 1\,\text{erg}\,\text{cm}^{-2}\,\text{s}^{-1}$, is converted to eddy energy by baroclinic instability or some other process (see Gill *et al.* 1974). Our findings thus provide a consistent link within the concept of an energy cascade from the general circulation through the eddy and internal-wave field down to the small-scale turbulence.

8. Conclusions

The interaction between short internal gravity waves and larger-scale motions in the ocean has been analysed theoretically. The analysis was based on the following premises.

(i) The small-scale field is a weakly nonlinear wave field. This enabled a rigorous closure of the equations of motion.

(ii) The wave field can be described adequately in the WKBJ approximation, implying the concept of propagating wave groups which transport momentum, buoyancy and energy. The validity of this concept restricts the large-scale flow to be quasi-geostrophic.

(iii) The cross-coupling between the wave field and the large-scale flow is weak. This assumption is not valid in regions of strong mean currents.

(iv) There exist relaxation processes within the wave field. These had to be considered so that the interaction leads to a secular energy exchange between the two fields.

The main results of the analysis are as follows.

(i) The mean flow causes a modulation of the internal-wave field which is a linear functional of the spatial gradients of the mean flow. This provides a signature of the interaction which is well suited for empirical tests.

(ii) The wave field also undergoes secular changes described by a Fokker–Planck equation. These changes have not been considered in detail.

(iii) The effect of the modulation on the mean flow reduces to a wave-induced diffusion of mean-flow momentum and buoyancy which is described by diffusion operators, or diffusion coefficients if relaxation processes are sufficiently strong. However, there is no vertical diffusion of buoyancy induced by propagating internal waves. The diffusion operators depend on the well-established frequency spectrum of the internal-wave field and rather insensitively on its relaxation time. Estimates of the effective diffusion coefficients lead to an exceptionally high value of the vertical viscosity coefficient, attributed to the fact that almost freely propagating wave groups transport momentum more effectively than eddies do.

The concept of nearly local diffusion operators is mainly applicable in the main thermocline. In the deep water column the diffusion is presumably non-local and requires the simultaneous consideration of the full water column. For the seasonal thermocline our arguments and estimates have to be reformulated since part of the internal-wave field is trapped. Our estimates of the wave-induced viscosity coefficients are consistent with the direct estimates by Frankignoul (1974, 1976). Other direct observational evidence is lacking. The discussion of the various implications did not bring forth any apparent inconsistencies, but suggested that the interaction between internal waves and meso-scale eddies might provide an important link between the energy input into the general circulation and the dissipation by small-scale processes.

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Appendix. Derivation of the diffusion operators N_h, N_v and K_h

The wave-induced momentum flux (6.1) can also be written as

$$F_{ij}^{(1)} = 2\int_f^{N_e} d\omega\, \mathrm{Re}\,\{\Gamma_{ij}^{(1)}(\omega)\}, \tag{A 1}$$

where

$$\Gamma_{ij}^{(1)}(\omega) = \tfrac{1}{2}\int d^3k\omega\delta(\omega-\Omega_0(\mathbf{k}))\, U_i^{(0)}U_j^{(0)*}D^{-1}\left[k_\alpha\frac{\partial}{\partial k_m}n^{(0)}\frac{\partial}{\partial x_m}\overline{u}_\alpha\right] \tag{A 2}$$

is the modulation of the cross-spectral matrix (5.13). From (2.26) we obtain explicitly

$$\begin{aligned} U_i^{(0)}U_j^{(0)*} = \alpha^2\beta^2k^{-2}\{&\beta^4\alpha^{-4}(\alpha_i\alpha_j+f^2\omega_0^{-2}\epsilon_{ik}\epsilon_{jm}\alpha_k\alpha_m)-\beta^2\alpha^{-2}(\alpha_i\beta_j+\alpha_j\beta_i)+\beta_i\beta_j\\ &+if\omega_0^{-1}\beta^4\alpha^{-4}(\epsilon_{ik}\alpha_k\alpha_j-\epsilon_{jk}\alpha_k\alpha_i)+if\omega_0^{-1}\beta^2\alpha^{-2}(\epsilon_{jk}\alpha_k\beta_i-\epsilon_{ik}\alpha_k\beta_j)\}, \end{aligned} \tag{A 3}$$

where $\boldsymbol{\alpha} = (k_1, k_2, 0)$ denotes the horizontal and $\boldsymbol{\beta} = (0, 0, k_3)$ the vertical wave-number vector. The integration over the direction of ϕ of the horizontal wave-number can be carried out in (A 2) since both the unperturbed action density $n^{(0)}$ and the operator D^{-1} are independent of ϕ. With

$$k_\alpha\,\partial n^{(0)}/\partial k_m = \alpha^{-1}\alpha_\alpha\alpha_m\,\partial n^{(0)}/\partial\alpha+\alpha_\alpha\delta_{m3}\,\partial n^{(0)}/\partial k_3, \tag{A 4}$$

the integrals involved are of the form

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, \alpha_i\,\alpha_j\,\alpha_k\,\alpha_m = \tfrac{1}{8}\alpha^4\,(\tilde{\delta}_{ij}\tilde{\delta}_{km}+\tilde{\delta}_{ik}\tilde{\delta}_{jm}+\tilde{\delta}_{im}\tilde{\delta}_{jk}), \tag{A 5a}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, \alpha_i\,\alpha_j\,\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} d\phi\,\alpha_i = 0, \tag{A 5b}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, \alpha_i\,\alpha_j = \tfrac{1}{2}\alpha^2\tilde{\delta}_{ij}. \tag{A 5c}$$

Here $\tilde{\delta}_{ij} = 1$ if $i = j = 1, 2$ and zero otherwise.

Collecting terms of the same structure we find

$$\begin{aligned} C_{ij}^{(1)} = \frac{1}{2}\int d^3k\omega\delta(\omega-\Omega_0(\mathbf{k}))\bigg\{&\frac{1}{8}\frac{\beta^2}{k^2}D^{-1}\left[\alpha\frac{\partial}{\partial\alpha}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right]\tilde{\delta}_{ij}\tilde{\delta}_{\alpha m}\\ &+\frac{3}{8}\frac{\beta^2}{k^2}\frac{f^2}{\omega_0^2}D^{-1}\left[\alpha\frac{\partial}{\partial\alpha}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right]\tilde{\delta}_{ij}\tilde{\delta}_{\alpha m}\\ &+\frac{1}{8}\frac{\beta^2}{k^2}\frac{\omega_0^2-f^2}{\omega_0^2}D^{-1}\left[\alpha\frac{\partial}{\partial\alpha}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right](\tilde{\delta}_{i\alpha}\tilde{\delta}_{jm}+\tilde{\delta}_{im}\tilde{\delta}_{j\alpha})\\ &-\frac{1}{2}\frac{\alpha k_3}{k^2}D^{-1}\left[\alpha\frac{\partial}{\partial k_3}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right](\delta_{j3}\tilde{\delta}_{i\alpha}+\delta_{i3}\tilde{\delta}_{j\alpha})\\ &+\frac{1}{2}\frac{\alpha^2}{k^2}D^{-1}\left[\alpha\frac{\partial}{\partial\alpha}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right]\delta_{i3}\,\delta_{j3}\,\tilde{\delta}_{\alpha m}\bigg\}, \end{aligned} \tag{A 6}$$

$$\begin{aligned} Q_{ij}^{(1)} = \frac{1}{2}\int d^3k\,\omega\delta(\omega-\Omega_0(\mathbf{k}))\bigg\{&\frac{1}{2}\frac{f}{\omega_0}\frac{\beta^2}{k^2}D^{-1}\left[\alpha\frac{\partial}{\partial\alpha}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right]\epsilon_{ji}\tilde{\delta}_{\alpha m}\\ &+\frac{1}{2}\frac{\alpha k_3}{k^2}\frac{f}{\omega_0}D^{-1}\left[\alpha\frac{\partial}{\partial k_3}n^{(0)}\frac{\partial}{\partial x_m}\bar{u}_\alpha\right]\delta_{m3}(\delta_{j3}\,\epsilon_{i\alpha}-\delta_{i3}\,\epsilon_{j\alpha})\bigg\}. \end{aligned} \tag{A 7}$$

For quasi-geostrophic flow ($\partial\bar{u}_\alpha/\partial x_\alpha = 0$) we can omit all terms proportional to $\tilde{\delta}_{\alpha m}$ and obtain

$$\left.\begin{matrix} C_{11}^{(1)}(\omega)\\ C_{12}^{(1)}(\omega)\\ C_{22}^{(1)}(\omega)\end{matrix}\right\} = -\tfrac{1}{2}N_h(\omega)\begin{cases} 2\partial\bar{u}_1/\partial x_1, & \text{(A 8a)}\\ \partial\bar{u}_2/\partial x_1+\partial\bar{u}_1/\partial x_2, & \text{(A 8b)}\\ 2\partial\bar{u}_2/\partial x_2, & \text{(A 8c)}\end{cases}$$

$$\left.\begin{matrix} C_{13}^{(1)}(\omega)\\ C_{23}^{(1)}(\omega)\end{matrix}\right\} = -\tfrac{1}{2}N_v(\omega)\begin{cases} \partial\bar{u}_1/\partial x_3, & \text{(A 8d)}\\ \partial\bar{u}_2/\partial x_3, & \text{(A 8e)}\end{cases}$$

$$\left.\begin{matrix} Q_{13}^{(1)}(\omega)\\ Q_{23}^{(1)}(\omega)\end{matrix}\right\} = -\tfrac{1}{2}N_0(\omega)\begin{cases} \partial\bar{u}_2/\partial x_3, & \text{(A 8f)}\\ -\partial\bar{u}_1/\partial x_3, & \text{(A 8g)}\end{cases}$$

$$C_{33}^{(1)}(\omega) = 0, \quad Q_{12}^{(1)}(\omega) = 0, \tag{A 8h, i}$$

implying the correlations listed in (5.14). The factors of proportionality are given by

$$\left.\begin{matrix} N_h(\omega)\\ N_v(\omega)\\ N_0(\omega)\end{matrix}\right\} = -\int d^3k\omega\delta(\omega-\Omega_0(\mathbf{k}))\begin{cases} \tfrac{1}{8}k_3^2(\omega_0^2-f^2)\,k^{-2}\omega_0^{-2}\,D^{-1}[k_\alpha\,\partial n^{(0)}/\partial k_\alpha\ldots], & \text{(A 9a)}\\ -\tfrac{1}{2}k_\alpha\,k_3 k^{-2}D^{-1}[k_\alpha\,\partial n^{(0)}/\partial k_3\ldots], & \text{(A 9b)}\\ \tfrac{1}{2}k_\alpha\,k_3 k^{-2}f\omega_0^{-1}\,D^{-1}[k_\alpha\,\partial n^{(0)}/\partial k_3\ldots]. & \text{(A 9c)}\end{cases}$$

For the divergence of the wave-induced momentum flux (A 1) we find

$$\frac{\partial}{\partial x_j} F_{ij}^{(1)} = -\frac{\partial}{\partial x_j} N_h \frac{\partial}{\partial x_m} \bar{u}_\alpha \tilde{\delta}_{i\alpha} \tilde{\delta}_{jm} - \frac{\partial}{\partial x_j} N_h \frac{\partial}{\partial x_m} \bar{u}_\alpha \tilde{\delta}_{im} \tilde{\delta}_{j\alpha}$$
$$- \frac{\partial}{\partial x_j} N_v \frac{\partial}{\partial x_m} \bar{u}_\alpha \delta_{m3} \delta_{j3} \tilde{\delta}_{i\alpha} - \frac{\partial}{\partial x_j} N_v \frac{\partial}{\partial x_m} \bar{u}_\alpha \delta_{m3} \delta_{i3} \tilde{\delta}_{j\alpha}, \quad \text{(A 10)}$$

where

$$N_h = \int_f^{N_e} d\omega \, N_h(\omega), \quad N_v = \int_f^{N_e} d\omega \, N_v(\omega). \quad \text{(A 11)}$$

Since N_h and N_v are independent of the horizontal space co-ordinate, the second and fourth term in (A 10) vanish and we obtain our results (6.3) and (6.4).

The wave-induced buoyancy flux is given by

$$M_\beta^{(1)} = \int d^3k \, \omega_0 \, \mathrm{Re}\{B^{(0)} U_\beta^{(0)*}\} D^{-1} \left[k_\alpha \frac{\partial}{\partial k_m} n^{(0)} \frac{\partial}{\partial x_m} \bar{u}_\alpha \right], \quad \text{(A 12)}$$

with

$$\mathrm{Re}\{B^{(0)} U_\beta^{(0)*}\} = -f \frac{N_e^2}{\omega_0^2} \frac{k_3 \alpha_j}{k^2} \epsilon_{j\beta}. \quad \text{(A 13)}$$

Integrating over ϕ we find

$$M_\beta^{(1)} = f \, K_h \frac{\partial \bar{u}_\alpha}{\partial x_m} \epsilon_{j\beta} \tilde{\delta}_{j\alpha} \delta_{m3} = -f K_h \epsilon_{\beta\alpha} \, \partial \bar{u}_\alpha / \partial x_3 \quad \text{(A 14)}$$

with

$$K_h[\ldots] = -\tfrac{1}{2} \int d^3k \, \omega_0^{-1} N_e^2 k_\alpha k_3 k^{-2} D^{-1} [k_\alpha \, \partial n^{(0)} / \partial k_3 \ldots], \quad \text{(A 15)}$$

which completes our results (6.3) and (6.4).

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